Guidelines for Preparing and Supporting Elementary Mathematics Specialists

Developed by the Association of Mathematics Teacher Educators

Guidelines for Preparing and Supporting Elementary Mathematics Specialists

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Preface

We are very excited to share *Guidelines for Preparing and Supporting Elementary Mathematics Specialists,* the Association of Mathematics Teacher Educators' (AMTE) most updated document that highlights the importance of these professionals and their specialized preparation. This document continues and expands recommendations from *Standards for Elementary Mathematics Specialists: A Reference for Teacher Credentialing and Degree Programs*, first published in 2009 and last updated in 2013. Advocating for effective mathematics teacher education is central to the mission of AMTE. For example, AMTE's new long-term goals for 2024-2028 include Goal 2, "support and provide guidance on the high-quality preparation, recruitment, retention, and diversification of mathematics teachers across the variety of educational spaces". This new document provides a way to enact this long-term goal. Preparing mathematics teachers, teacher leaders, and coaches at all levels, especially PK-6, is a critical need and the original 2009 document was designed to inform ways to effectively prepare Elementary Mathematics Specialists. In 2013, a revised version was published to reflect recommendations from the Conference Board of Mathematical Sciences' *The Mathematical Education of Teachers II* (2010), the National Council of Teachers of Mathematics' Elementary Mathematics Specialist Standards (2012), as well as the Common Core State Standards - Mathematics (2010). This current document reflects the research produced since the publication of the first standards and recommendations from AMTE's *Standards for Preparing Teachers of Mathematics* (2017). It also includes descriptions, examples, and vignettes to make the intentions of the guidelines come alive.

In 2022, AMTE, along with ASSM, NCSM, and NCTM, published an updated joint position statement about the role of Elementary Mathematics Specialists (EMSs), which affirmed a collective commitment to using these professionals in PK-6 environments to enhance the planning, teaching, learning, and assessment of mathematics. The preparation of the position statement highlighted the need for updating the *Standards for Elementary Mathematics Specialists: A Reference for Teacher Credentialing and Degree Programs* (AMTE, 2013) to address emergent issues related to technology and equity, AMTE's *Standards for Preparing Teachers of Mathematics* (2017), and current research. Thus, an EMS Standards Task Force was established to complete this challenging work, which is shown below.

Co-chair: Nicole Rigelman, Portland State University and The Math Learning Center
Co-chair: Susan Swars Auslander, The University of Alabama
Skip Fennell, McDaniel College
Marta Garcia, Mt. Holyoke and TERC
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Kate Roscioli, George Mason University and Prince William County Public Schools

AMTE would like to thank the chairs who led this important task force and the members who supported the development of these important updated and expanded guidelines. The chairs and members dedicated countless hours and energy to produce this important document because they recognize its importance. AMTE also thanks invited reviewers and AMTE members who provided feedback on a draft of this document.

We hope that you find this document useful for informing, advocating, and revising practices, structures, and policies regarding Elementary Mathematics Specialists. May this document foster change that further informs and advances mathematical teaching and learning.

Enrique Galindo, AMTE President 2023-2025
Megan Burton, AMTE President 2021-2023

Introduction

In 2022, the Association of Mathematics Teacher Educators (AMTE), the Association of State Supervisors of Mathematics (ASSM), the National Council of Supervisors of Mathematics: Leadership in Mathematics Education (NCSM), and the National Council of Teachers of Mathematics (NCTM) published an updated call for Elementary Mathematics Specialists (EMSs) to support high-quality teaching of mathematics and to ensure equitable and effective mathematics learning for each and every student. These organizations reiterated the goal stated 13 years earlier in their first joint position statement of **every elementary school having access to an EMS professional**.[1]

The research base on the value of EMSs for students and teachers has grown significantly since the release of AMTE's *Elementary Mathematics Specialist Standards* in 2009,[2] securing EMSs' important role in strengthening mathematics programs in schools and districts. Whether formally leading in the role of a mathematics coach or informally leading from the classroom,[3] EMSs collaborate with colleagues either one-on-one or in grade-level teams to support shifts toward ambitious and equitable mathematics teaching practices[4] in every classroom. These research-informed teaching practices provide opportunities for all students to learn mathematics through: deep engagement with the content and practices and processes; collaborative discussion and debate of their mathematical ideas with one another; and affirmation and leveraging of their diversities and mathematical strengths.

Who Are Elementary Mathematics Specialists?

Elementary Mathematics Specialists (EMSs) are informal or formal teacher leaders responsible for supporting ambitious and equitable mathematics instruction and improving student learning experiences at the classroom, school, district, county, or state levels. A specialist's specific roles and responsibilities vary according to the needs and plans of each setting. An EMS may work with educators one-on-one (e.g., collaborating, modeling, co-teaching, coaching, mentoring) or in groups providing mathematics professional learning and support (e.g., to grade level teams, across grade teams, school faculty, district groups,

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[1] AMTE et al., 2022; McGatha et al., 2015.

[2] AMTE, 2010. The *EMS Standards* were revised in 2013 in response to the release and widespread adoption of the *Common Core State Standards for School Mathematics* (National Governors Association and the Council of Chief State School Officers, 2010).

[3] McGatha and Rigelman (2017) provide a framework for describing roles of EMSs to include: elementary mathematics teachers, teaching math to more than one group of students; elementary mathematics interventionists, teaching math to students who may benefit from more support or further challenge; elementary mathematics coaches, working with adults to develop their mathematics content and pedagogical knowledge as well as their strengthen their mathematics identity and agency.

[4] This phrase is used in alignment with the mathematics teaching practices advocated in *Catalyzing Change in Early Childhood and Elementary Mathematics* (NCTM, 2020) and other published positions of NCTM.

administrators). An EMS may work with students as a *generalist* teaching all subjects, a *specialist* teaching mathematics to multiple groups of students, or an *interventionist* teaching mathematics to groups of students who benefit from more time or support or those who benefit from additional challenge. As indicated in Figure 1, EMSs may have a combination of student-facing (e.g., specialist or generalist teacher, interventionist) and adult-facing (e.g., grade-level mathematics leader or coach, school-level mathematics leader or coach, district-level mathematics leader or coach) roles. While the figure is not inclusive of every potential EMS role, configuration, or title, it is intended to communicate the range of support available for students and teachers and offered by these informal and formal teacher leaders. Further, in their work, EMSs are *influencers* supporting families and communities to understand approaches taken to mathematics instruction as well as serving on committees to develop curriculum, assessments, and policies concerned with mathematics education. Whatever the setting or responsibilities, EMSs need a deep and broad knowledge of mathematics content, expertise in using and helping others use ambitious and equitable teaching practices, and the ability to lead and support efforts that help each and every student learn important mathematics.

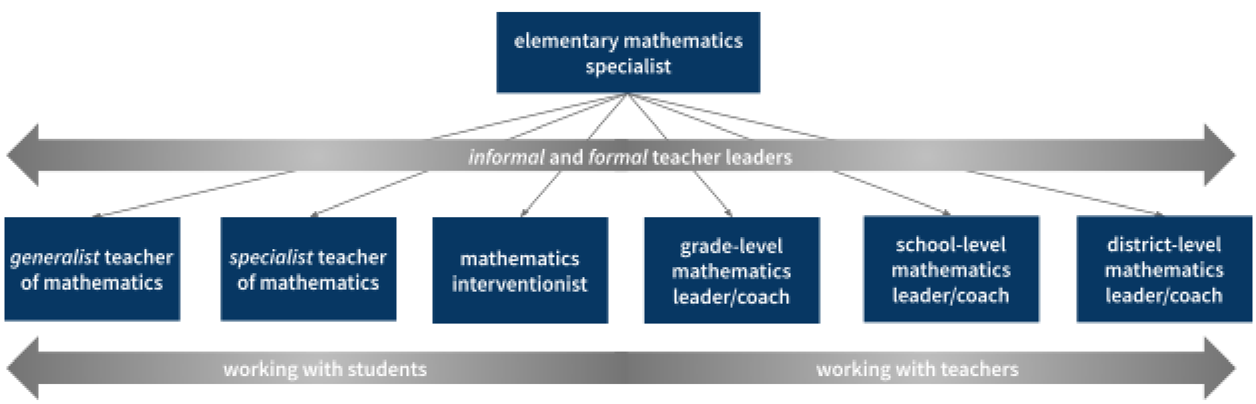

Figure 1
Leadership Roles and Responsibilities of Elementary Mathematics Specialists[5]

Purpose

Because most elementary teachers are prepared as generalists, those interested in serving as an EMS (i.e., generalist or specialist teacher, interventionist, leader, coach) need additional study of mathematics content for teachers and responsive mathematics pedagogy. As formal and informal teacher leaders, EMSs also must understand the unique learning needs of adults, strategies for supporting collaborative and transformative professional learning, and approaches to advocacy on behalf of students, teachers, mathematics, and programs.

These guidelines detail the mathematics content, mathematics pedagogy, and mathematics

[5] Rigelman et al., 2024.

leadership knowledge, skills, and dispositions necessary for the various roles and responsibilities an EMS may assume. The guidelines not only serve as a response to two of the action steps recommended in the recent joint position statement on EMSs[6] (see Table 1), but more importantly, they provide a vision for the preparation and ongoing support for EMSs regardless of where that preparation and support takes place (e.g., universities, regions, school districts). Table 1 displays the two relevant action steps alongside ways the guidelines could be used to support these action steps, and the rationale linking the potential uses with the action steps.

Table 1
Potential Uses of the *Guidelines for Preparing and Supporting Elementary Mathematics Specialists*

Proposed Actions[7] and Rationales	**Potential Uses of these Guidelines**
Action Collaborate to establish incentives for EMSs in recognition of the positive impact they can have on the day-to-day work in schools **Rationale** Licensure pathways elevate the EMS roles and the importance of specialized preparation for their work	State and national teacher-licensing agencies (i.e., National Board for Professional Teaching Standards)[8] use these standards to define expectations when developing or revising the credential. Testing companies use these standards to develop professional exams.
Action Provide high-quality mathematics professional learning that not only prepares EMSs for their work but also supports and sustains district and school ongoing mathematics improvement efforts **Rationale** Recognition of a 1) novice-expert continuum for informal and formal teacher leaders, and 2) need for ongoing learning of EMSs just as other educators have this need	Institutions of higher education or other agencies building local teacher leadership capacity use these guidelines to provide definition and focus for the mathematical content, pedagogy, and leadership preparation of EMSs. Schools, districts, or states with EMSs in their educator workforce use these guidelines to design ongoing EMS professional learning and support.

Finally, these guidelines are informed by prior articulations of: the knowledge needed by PK-8

[6] AMTE et al., 2022.
[7] The proposed actions are drawn from the Action Steps in the joint position statement, see AMTE et al., 2022.
[8] At the time of this writing, there are 25 states with a state-level EMS license, certificate, or endorsement that can be added to a teaching license, and there is no national elementary mathematics-focused certification. This is in contrast to every state and the National Board for Professional Teaching Standards offering a reading specialist credential.

students in mathematics; the knowledge needed for ambitious and equitable teaching of elementary mathematics; and the leadership and coaching knowledge domains for mathematics as well as for teaching more broadly. For the full list of standards and policy documents influencing these guidelines, go to: http://amte.net/ems.

Audience

Because there are many teachers across the U.S. being called upon to fill specialized elementary mathematics teaching and leading roles and often without focused training, the audience for this document is any mathematics teacher educator working to support advanced elementary mathematics preparation. These Elementary Mathematics Specialist teacher educators (EMSTEs) support EMSs who lead, or are interested in leading, from their classrooms, within grade-level teams, and across their schools or districts. EMSTEs may work individually or collaboratively at the district, state, or higher education level and have a role in providing high-quality mathematics professional learning that not only prepares EMSs for their work, but also supports and sustains their district- and school-wide ongoing mathematics improvement efforts.

When designing learning opportunities for EMSs, EMSTEs should consider EMSs' existing preparation in mathematics content, pedagogy, and leadership, as well as the expertise acquired from teaching and leading experiences. This information should be used in determining the duration and proportion of time focused on developing content, pedagogy, leadership, or the integration of related knowledge, skills, and dispositions (e.g., content-focused pedagogy courses).

Organization of The Document

The document is divided into three sections by domain: Mathematics Content, Mathematics Pedagogy, and Mathematics Leadership. Listed below are the sections and the associated standards.

Mathematics Content: Knowledge, Skills, and Dispositions

- C.1. Understanding Number Relationships and Structure
- C.2. Generalizing Behaviors of Operations across Number Domains
- C.3. Recognizing, Extending, and Generalizing Mathematical and Everyday Patterns
- C.4. Exploring, Examining, and Enumerating Space
- C.5. Investigating Questions and Interrogating Data through Statistical Problem Solving

Mathematics Pedagogy: Knowledge, Skills, and Dispositions

- P.1. Knowing Students to Foster Positive Mathematics Identity
- P.2. Planning for Responsive Instruction and Making Curricular Decisions

P.3. Implementing Ambitious and Equitable Mathematics Instruction
P.4. Assessing Student Understanding and Learning

Mathematics Leadership: Knowledge, Skills, and Dispositions

L.1. Advocating for Ambitious Instruction and Equitable Structures for Students and Teachers
L.2. Advancing Implementation of Ambitious and Equitable Mathematics Instruction and Assessment
L.3. Activating Continuous Mathematics Professional Learning and Program Improvement
L.4. Developing and Sustaining a Culture of Collaboration to Support Mathematics Teaching and Learning

Each section's introduction provides details regarding the structure and features, followed by a list of the standards and indicators. The balance of the section provides descriptions and examples to illuminate the intentions of each standard and indicator. The pedagogy and leadership sections also include vignettes reflecting the broad range of EMS roles and the integrated nature of EMS work as shown with tags to relevant indicators. EMSTEs will likely find it useful to draw upon standards from a single domain or from across domains (e.g., content and pedagogy, pedagogy and leadership) due to the integrated nature of the domains as they design EMS professional learning or university courses. They should also consider how to prepare and support EMSs as they expand their identity from a role of predominantly supporting student learning to one that also includes supporting collegial learning of educators and mathematics program improvement.[9]

Certification Programs

Currently, almost half of the states in the U.S. offer a pathway for EMS licensure, certification, or endorsement. However, programs designed to prepare these professionals differ in substantive ways. Some are designed to transition elementary-certified teachers to middle school mathematics teaching assignments. Some focus on broadening and deepening the content and pedagogical knowledge of elementary teachers of mathematics. Others prepare elementary teachers for leadership or coaching responsibilities and focus on a combination of mathematics content, pedagogy, and leadership.

The following are recommendations for college- or university-based programs that prepare EMSs and emphasize a combination of mathematics content, pedagogy, and leadership.

Prerequisites: Teacher certification and at least 3 years of successful mathematics teaching experience.

[9] Chval et al., 2010; Knapp, 2017.

Program of Study: To include

- An appropriate proportion of each area: content, pedagogy, and leadership equivalent to at least 18 semester-hours or 27 quarter-hours. EMS programs may be a stand-alone series of courses potentially leading to an endorsement or certificate, or these courses may be part of an advanced degree program with additional credit requirements.
- A supervised mathematics leadership practicum in which a candidate acquires experience supporting a range of student and adult learners, including:
  - elementary students (e.g., primary, intermediate, multilingual, neurodiverse); and
  - elementary school educators (e.g., novice, experienced, specialists, paraeducators) in a variety of formal and informal professional learning settings.

For a full list of resources to support development of EMS programs and ongoing professional learning, go to: http://amte.net/ems.

Mathematics Content: Knowledge, Skills, and Dispositions

Elementary Mathematics Specialists (EMSs) may serve as teachers as well as teacher leaders in their settings. In their roles, EMSs may facilitate mathematics learning for PK-6 students (e.g., classroom teachers, specialist teachers, interventionists) as well as for teachers (e.g., informal leaders, grade-level coaches, building- or district-level coaches). The preparation and ongoing support for the mathematical knowledge, skills, and dispositions necessary for these positions are at a broader and deeper level than the mathematical content expectations for those prepared as multi-subject elementary teachers.[10] Notably, there must be explicit attention to the mathematical expectations at the early childhood and elementary levels as well as at the middle school and early high school levels. EMSs must know what elementary students need to understand and be able to do as well as how future learning builds upon that foundation. EMSs also need rich, well-connected understandings of the specialized knowledge needed for teaching mathematics and how that knowledge intersects with: (1) student knowledge and dispositions, the development of ideas from in- and out-of-school experiences, and typical learning progressions and challenges; and (2) teacher knowledge and dispositions, state and nationally recommended content and practice and process standards, and the development of the mathematical storyline across adopted standards and instructional materials. EMSs' mathematical content knowledge, skills, and dispositions should be regularly updated and deepened to support continuous learning and improvement for each and every student and teacher influenced through their role.

Each standard within the mathematics content domain has a common structure that includes a description of the important work of the standard and a justification for inclusion of the standard. Listed as a sidebar are the indicators for the standard. Each indicator generally follows a three-part structure: (1) connects knowledge of content and the intuitive and informal knowledge children bring to school; (2) connects knowledge of content and the school-focused experiences that support knowledge development within the domain; and (3) connects knowledge of content and the ways EMSs use their deep, well-connected knowledge in support of PK–6 student and teacher learning of mathematics. Additional sidebars convey the integrated nature of the mathematics content and the practices and processes.[11] Rather than simply providing a list of the disciplinary practices, the sidebars provide an image of productive, authentic engagement in the practices and processes as a means of learning content. These are examples of possible integrations and are not intended to represent all the possible content-practice connections. Finally, given the importance of horizon content knowledge (i.e., "awareness of how mathematical topics are related over the mathematics

[10] Association of Mathematics Teacher Educators [AMTE], 2017; Conference Board of the Mathematical Sciences [CBMS], 2012.

[11] *Catalyzing Change in Early Childhood and Elementary Mathematics* (NCTM, 2020) advocates for the use of these four interrelated practices and processes to engage each and every student as *doers* of mathematics: (1) representing and connecting, (2) explaining and justifying, (3) contextualizing and decontextualizing, and (4) noticing and using mathematical structures.

span included in the curriculum"[12]) and across-the-grades coherence to EMSs' work, each standard section concludes with a table that conveys the foundational concepts and how they build across the grades from Kindergarten through grade 9. It should be noted that rather than these standards providing an exhaustive list of mathematical topics, they include critical topics for EMSs to know and understand, with specific attention to student thinking, learning, and teaching.

[12] Ball, 1993; Ball et al., 2008, p. 403.

Standard C.1. Understanding Number Relationships and Structure

C.1.a. Building connections between counting and cardinal understanding of number

C.1.b. Making sense of number composition and the base ten number system

C.1.c. Understanding fractions and decimals as an extension of the number system

Standard C.2. Generalizing Behaviors of Operations across Number Domains

C.2.a. Building understandings of the operations through solving contextualized situations

C.2.b. Fostering connections among the operations

C.2.c. Extending understandings of the operations to multi-digit whole numbers

C.2.d. Extending understandings of the operations to all rational numbers

Standard C.3. Recognizing, Extending, and Generalizing Mathematical and Everyday Patterns

C.3.a. Recognizing, extending, and making generalizations about repeating and growing patterns

C.3.b. Representing functions with visual models and contextual situations

Standard C.4. Exploring, Examining, and Enumerating Space

C.4.a. Recognizing, naming, describing, and comparing shapes

C.4.b. Composing, decomposing, and understanding space

C.4.c. Understanding spatial relationships and spatial structuring

C.4.d. Enumerating space through geometric measurement

Standard C.5. Investigating Questions and Interrogating Data through Statistical Problem Solving

C.5.a. Collecting, organizing, and representing data

C.5.b. Selecting and using appropriate statistical methods to analyze data

C.5.c. Interpreting data: developing inferences and evaluating predictions

C.5.d. Understanding basic concepts of probability

Standard C.1. Understanding Number Relationships and Structure

C.1.a. Building connections between counting and cardinal understanding of number

C.1.b. Making sense of number composition and the base ten number system

C.1.c. Understanding fractions and decimals as an extension of the number system

Developing a rich, flexible understanding of what numbers are, how they can be combined and separated in different ways, their relative magnitude and relationship to benchmark numbers, and the structure of our base ten number system is a central goal of learning and teaching mathematics at the elementary school level.[13] This standard focuses on developing robust understandings of numbers and the number system. Supporting the development of number and number sense has far-reaching implications for learning, with research showing that young children's understanding of counting and early number are the strongest predictors of fifth-grade mathematics achievement.[14]

Elementary Mathematics Specialists (EMSs) understand the fundamental role of counting in making sense of number and in providing a foundation for understanding the operations. Counting also makes it possible to compare and order quantities, and to derive meaning of the base ten number system. As the number system is extended to include rational numbers, EMSs understand that fractions and decimals are also numbers and composed of countable units, allowing for additional work with counting, comparing, ordering, and relating these amounts to benchmarks and estimation. EMSs work to build robust and flexible understandings of what numbers are throughout the elementary grades by supporting coherence across counting, the base ten number system, and rational numbers.

C.1.a. Building connections between counting and cardinal understanding of number

Counting provides a foundation for children to make sense of what numbers are and how they relate to one another. While among the first mathematics opportunities with which young children are likely to encounter and engage, counting objects and understanding the unit is a complex, interconnected process. For young children, learning to count involves learning to coordinate each individual object or thing being counted with the sequence of number words, and coming to understand that the final number assigned when counting reflects a particular property of the entire set—its quantity or cardinal value. Even very young children often display an emerging, intuitive sense of fundamental counting principles. For example, without being instructed to do so, children will use a variety of methods for keeping track of their counting so that each object is counted once and only once (e.g., by standing up objects as they count, moving objects, or lining them up). Later, as they attempt to count beyond ten,

[13] National Council of Teachers of Mathematics [NCTM], 2000.
[14] Nguyen et al., 2016.

children's invented number words (five-teen; twenty-ten, twenty-eleven) and sequences (28, 29… 40, 41, 42) often reveal an awareness of the patterns and underlying structure of the base ten number system. Regular and varied opportunities to count collections of objects allow children to grapple with and make meaning of the relationship between the process of counting and its outcome. Allowing children to count in ways that they choose (e.g., rather than requiring that children place objects in a line before counting) supports them in developing flexibility when determining what must always be the same when counting (the *stable order* of the number word sequence) and what can be altered (objects can be counted in whatever order you wish–*order irrelevance*). Collections provide opportunities to compare and reason about subsets of objects (e.g., "Do you have more yellow bears or blue bears?"), and can provide an entry point into problem solving (e.g., "If all the green bears walked away, how many bears would you have then?"). Counting progressively larger collections provides a natural invitation to organize and create groups, supporting children to connect skip counting sequences with the equal groups they represent.[15]

Building deep understanding of how counting relates to number is a central theme and foundational element of elementary mathematics. In essence, numbers themselves are "ideas—abstractions that apply to a broad range of real or imagined situations."[16] Numbers are encountered in different contexts that reflect different meanings. Most often, numbers are used to indicate amounts, or cardinal values (e.g., three cookies, 15 minutes). However, numbers can also hold ordinal (e.g., third in line, second floor) or nominal (uniform number 22, 6475 Alvarado Road) meanings. Cardinal understanding allows children to use the ordinal nature of the number word sequence to reason about number relationships. For example, quantities can be compared by counting each set or by knowing what number comes before or after another number in the sequence (e.g., six is more than four because it comes after). Cardinal understanding also allows children to make sense of how, when counting: (a) the next number is one more than the previous number (e.g., 6 is one more than 5–sometimes called the *successor function*); and (b) that each new number also includes the previous quantities in the sequence (e.g., there is a 4 inside of 5–sometimes referred to as *hierarchical inclusion*).

The mathematical relationships, processes, and practices children engage in through counting are developed and expanded upon throughout the elementary grades and beyond. EMSs understand that learning to count with understanding is complex. Children's grasp of the number word sequence, one-to-one correspondence, and cardinality develop concurrently rather than sequentially and in different ways for different children. EMSs understand the ways in which counting and cardinality provide a foundation for the operations and how they relate to one another. For example, creating, acting upon, and counting sets of objects allow children to begin to represent and solve many different kinds of problems. EMSs support the use of objects, fingers, sounds/actions, tallies, and drawings and

[15] Carpenter et al., 2017; Franke et al., 2018.
[16] National Research Council [NRC], 2001, p. 72.

recognize the importance of connecting written methods to manipulatives and the actions of counting and representing collections. They understand that extending the number sequence beyond single digits involves making use of the base ten structure of the number system as well as correspondences between units. Counting ideas will be extended to counting by groups (e.g., 2s, 5s, 10s), unit fractions, and decimals.

C.1.b. Making sense of number composition and the base ten number system

***Noticing and Using Mathematical Structures**. When students have opportunities to consider varied ways of grouping collections, and recording the results, they notice the special relationship between the number of total units and the number of groups and number of ones when creating groups of ten. For example 89 units can be grouped by fives totalling 17 groups of 5 and 4 ones; or groups by twos totalling 44 groups of 2 and 1 one; but when grouping by tens there are 8 tens and 9 ones, corresponding to the 89 units.*

At the same time children build connections between counting and number, they begin to notice how amounts are grouped together in everyday life (e.g., my hand has 5 fingers, there are 8 crayons in 1 box), and how quantities can be thought of as composed of smaller groups. For example, a child might remark that within an arrangement of 5 blueberries on a plate, there is a set of 2 blueberries and another set of 3 blueberries. This ability to perceive small quantities as total units in and of themselves (without counting) is called *subitizing*.[17] Flexibility in composing and decomposing quantities in different ways (e.g., 5 = 2 + 3 = 1 + 4 = 1 + 2 + 2) is essential in allowing children to make use of the structure of numbers and the number system to reason about and solve problems. A critical phase of development is when children begin to connect the patterns of the number word sequence with the base ten structure of the number system. For example, children are often able to count using the final numbers in the teen sequence (e.g., 16 through 19 where the naming structure is more transparent, but omitting 13 or 15) before accurately using the entire sequence of teen number words.[18] Similar to other numbers, at first children understand 10 as composed of 10 ones. Over time, however, children will begin to work additively with 10 as a unit in and of itself, counting by 10s or adding and subtracting 10 at a time. Eventually, children begin to understand that the number names (and written numerals) reflect counts of different base ten units (89 is 8 tens and 9 ones) and use this understanding of the base ten structure of the number system to reason about and solve problems.[19]

During elementary school, work with number shifts from working with collections of individual units (24 = 1 + 1 + 1 + 1…), to additive compositions of different parts numbers (24 =

[17] Clements, 1999.
[18] Gould, 2017; McMillan et al., 2023.
[19] Carpenter, et al., 2015; Jaslow & Jacobs, 2009.

12 + 12 or 10 + 10 + 4), to multiplicative compositions (24 = 4 × 6) or a combination of the two (24 = (2 × 10) + 4). Similarly, models and representations of these ideas shift from concrete representations of individual objects (e.g., counters or drawings), to composing or decomposing numerals, to using formal mathematical symbols within expressions and equations. In making connections between these varied representations of number, it is critical to understand the equal sign as denoting a relationship of equivalence, rather than merely signaling the outcome of performing an operation. Overcoming the ways children and adults are conditioned to expect equations in the form of a + b = c often requires explicit conversations and consistent use of the conventions of what is "allowed" in equation formats when using the equal sign.[20] The idea that the value of a number corresponds with the sum of the base ten value of each digit is central to this work. Within whole numbers, the value of the place farthest to the right is one; the value of all other places is ten times the value of the place to its right. This idea is then extended to decimal numbers when numbers can be described as being $\frac{1}{10}$, $\frac{1}{100}$, or $\frac{1}{10^n}$ of a whole. Patterns in the number of zeros of the product occur when multiplying a whole number by powers of 10. Similarly, patterns in the placement of the decimal point occur when a decimal is multiplied or divided by a power of 10. Student generalizations that "you can add zeros" to whole numbers or that "the decimal point moves," in one direction or the other while appearing to be a mathematical shortcut, can emerge from students as a conjecture and be defended mathematically, through problem solving. It should not be taught as a rule.

EMSs understand that the idea of ten ones being simultaneously thought of as one ten is a complex idea. They recognize the importance of experiences that allow students to represent the base ten structure of numbers (e.g., piles of ten, ten frames, cubes connected in towers of ten) before they work with pre-structured proportional representations (e.g., base ten blocks). EMSs understand that the relationship between places in a multi-digit number is multiplicative, and that the multiplicative reasoning needed to understand base ten relationships develops over time. EMSs also understand that students often begin working with the principles of place value before they learn to multiply, and that understanding connections between multiplication by a multiple of ten and place value can deepen the sense making of place value. It is critical that EMSs recognize what is right about a child's thinking, even when they express their ideas in ways that might at first appear to be incorrect or incomplete. For example, an EMS considers the base ten understanding of a child who writes 100604 after counting a collection of 164 colored popsicle sticks knowing that written and spoken numbers are different and related systems. They understand zero is complex and has a role in place value (i.e., as a placeholder) that is different from the roles zero has in the number system (i.e., the boundary between negative and positive numbers) and computation (e.g., identity element for addition: zero added to any number results in that number; zero property: any number multiplied by zero equals zero). They recognize that as students consider the regularity of the progression of products when multiplying by powers of ten (for

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[20] Carpenter et al., 2003; Rittle-Johnson et al., 2011.

example 53, 530, 5,300, etc.), and describe and explain why this regularity occurs, students are extending the ideas of the base ten structure to larger numbers. EMSs support work in this area that goes beyond rote understandings that simply involve stating “the number of tens'' and “the number of ones” (e.g., understanding that the number of tens in 385 is not 8 but is 38 tens). They understand the place-value system is based on powers of ten and multi-digit numbers can be expressed using expanded notation with the digits 0 through 9 multiplied by the appropriate power of 10 and connected to a count of base ten blocks. With this knowledge, EMSs develop strategies for engaging others in understanding the multiple ways a number can be decomposed and how conceptual understanding precedes computational proficiency.[21]

C.1.c. Understanding fractions and decimals as an extension of the number system

Young children can talk about, represent, and reason about fractional amounts prior to formal instruction. In children’s everyday experiences, the word half might refer informally to parts or pieces of something that was once whole (e.g., “I broke the graham cracker into 3 halves”), or to a set (e.g., “so far I’ve only eaten half of my candy”). Children may also associate terms such as *quarters* with particular amounts (i.e., a quarter is 25 cents, but a quarter hour is 15 minutes) rather than as a fractional relationship to a whole (i.e., one-fourth of the whole). However, children are able to build from these everyday ideas despite their informality and lack of precision. Young children can tell you they would rather be a child sitting at the table where three people are sharing one cookie than at the table where four people are sharing one cookie. They can also begin to create drawings that represent fractional quantities through solving problems involving whole numbers where the remainders can be partitioned or shared (see the operations section). Children will often describe the amounts they create using informal language. For example, when asked how 4 people could share 10 submarine sandwiches, a child split each sandwich into four parts (so that each person would get $\frac{1}{4}$ of each sandwich), but described her answer as “10 pieces” (see figure 2). Equal sharing tasks like this example allow children to build from what they know about a whole to begin to represent and describe fractional amounts, providing a foundation for introducing formal fraction terms and symbols.[22] They also provide opportunities for children to wrestle with counting and combining unit fractions ($\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}= 10 \times \frac{1}{4} = \frac{10}{4}$) or to explore questions about equivalence that emerge from comparing different solution paths (e.g., is $\frac{10}{4}$ the same as two whole sandwiches and $\frac{1}{2}$ of a sandwich?).

[21] NCTM, 2023a.
[22] Empson & Levi, 2011*;* Siegler et al., 2010.

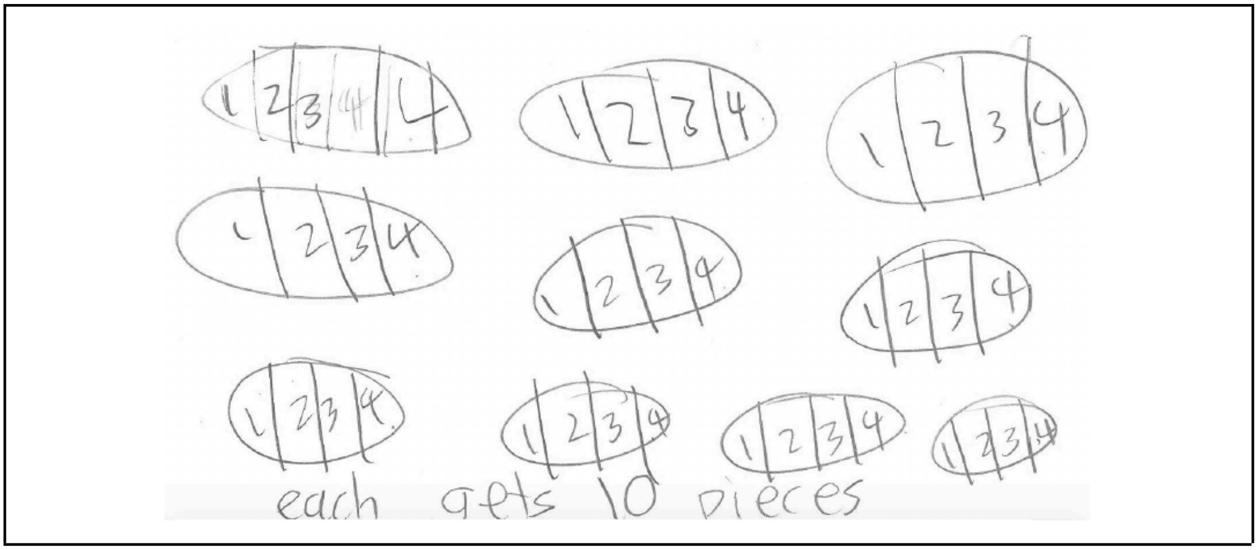

Figure 2
Child's Solution for How 4 People Could Share 10 Submarine Sandwiches

Fractions can hold different meanings depending on the context in which they are used.[23] A critical concept that underlies all of these meanings is that fractions are numbers. Fractions can be used to describe an amount that reflects a *part-whole* relationship ($\frac{3}{4}$ of a brownie). Fractions can also describe the result of division or a *quotient* (when 4 people share 3 brownies equally, each person receives $\frac{3}{4}$ of a brownie). A *ratio* interpretation of a fraction could describe the relationship between types of brownies (at the party there are 3 cream cheese brownies for every 4 chocolate chip brownies) or between a type of brownie and the total number of brownies (there are 3 cream cheese brownies for every 4 brownies), with only the later equivalent to $\frac{3}{4}$ when interpreted using different meanings involving the same sized whole. Fractions, ratios, and rates are connected by unit rates, with unit rates being used to solve problems and formulate equations for proportional relationships.[24] Fractions can describe *measures*, such as $\frac{3}{4}$ cup of chocolate chips for each batch of brownies. Finally, fractions can be used as *operators*, indicating that an amount is scaled or changed in a particular way (e.g., if you only want to make $\frac{3}{4}$ of a batch, take $\frac{3}{4}$ of the amount listed for each ingredient). In each of these interpretations, defining what constitutes the whole in a given problem or situation is fundamental to the meaning of fractions. For example, consider a situation where there are 2 boxes of granola bars with 4 granola bars in each box. The meaning of $\frac{1}{4}$ is dependent upon what is identified as the whole. If the whole is both boxes, $\frac{1}{4}$ would mean 2 granola bars. If the whole is one box, $\frac{1}{4}$ would mean 1 granola bar. If the whole is one granola bar, $\frac{1}{4}$ would mean $\frac{1}{4}$ of a granola bar.

Representing and Connecting.
As students explore the multiple meanings of fractions in varied contexts, it can be useful to examine and make connections to what both the numerator and denominator mean within and across a particular context. For a part-whole interpretation, $\frac{3}{4}$ of a brownie suggests a brownie has been cut into 4 equal parts (denominator) and three of the parts are counted (numerator). For a quotient interpretation, $\frac{3}{4}$ is the size of the resulting piece when 3 brownies (numerator) were cut into 4 parts (denominator). In the case of the ratio interpretation, the 3 is the count of a particular brownie type while the meaning of the denominator changes depending on whether the ratio is part-part or part-whole.

While student intuition is powerful for early understandings and making meaning of rational numbers, representing them symbolically is complex and often counterintuitive. For example, one-fourth can be represented symbolically in multiple ways as a fraction in the form $\frac{a}{b}$ ($\frac{1}{4}$,

[23] Barnett-Clarke et al., 2010; Behr et al., 1983; Kieren, 1992; NRC, 2005.
[24] CBMS, 2012.

$\frac{2}{8}$, $\frac{25}{100}$) or as a decimal fraction (0.25, 0.250, 0.25000000). EMSs recognize that earlier work with whole numbers can be overgeneralized (e.g., 4 has a greater value than 2 so $\frac{1}{4}$ must have a greater value than $\frac{1}{2}$) so students need opportunities to work with a variety of representations of fractions. These include linear, set, and region models to help students develop understandings of fractions as numbers. Additionally and importantly, just as students use manipulative materials and drawings to help anchor a mental image of a whole number, they can use the number line to show how a fraction (or decimal or percent) can be inserted between any two fractions. The number line provides a visual representation of fractions and decimals, and also serves as a measurement or iteration model for computation.[25] This visual representation also supports "seeing" the density of rational numbers, which is particularly important at the middle school level and beyond.[26] EMSs understand how fraction equivalence and magnitude, that is being able to determine if a number is greater or less than another number (e.g., $\frac{4}{5}$ is close to but less than 1), is foundational for extending work with rational numbers. Developing flexibility in creating equivalent fractions and using fractions such as $\frac{1}{2}$ or $\frac{1}{4}$ as "benchmarks" allows learners to informally extend their fraction understandings to include comparing and ordering fractions.[27] This flexibility should be extended to representing, comparing, and ordering equivalent fractions, decimals, and common percents (e.g., $\frac{1}{2} = \frac{5}{10} = 0.5 = 50\%$, $0.9 > \frac{7}{8}$) at the appropriate grade levels.

[25] Fennell, 2007.
[26] Saxe et.al., 2007.
[27] Bray & Abreu-Sanchez, 2010.

Figure 3

Mathematical Storyline for Number Relationships and Structure

PK-Grade 1	Grades 2-3	Grades 4-5	Grades 6-7	Grades 8-9
Understand the relationship between numbers and quantities	Understand fractions as numbers that represent part of a whole or a measurement	Understand fractions as division and as an operator	Understand fractions as a ratio	Understand connections between proportional and linear relationships
Extend understanding of number to larger numbers as groups of 10s and 1s		Extend understanding of the number system to include all positive rational numbers	Extend understanding of the number system to include negative rational numbers	Extend understanding of the number system to all real numbers
Compose and decompose quantities to recognize cardinality of small groups (i.e., subitize) and to support comparison and computation	Compose and decompose whole numbers by place value to support comparison and computation	Compose and decompose whole numbers up to a million and decimals to thousandths by place value to support comparison and computation	Develop a unified understanding of number, recognizing fractions, finite and repeating decimals, and percents as different representations of rational numbers	Develop a unified understanding of rational and irrational numbers and develop strategies for using rational approximations of irrational numbers to compare irrational numbers
Recognize quantities are the same regardless of arrangement or order of the count	Recognize that quantities can be named in different ways.	Recognize and create equivalent fractions as equal portions of the same size whole	Recognize equivalent ratios as proportions	Recognize that relationships of two quantities can be described in terms of ratios, rates, percents, or proportional relationships
Count by 1s or by groups of 5s and 10s to answer "how many?"	Count by unit fractions to answer "how many?" or "how much of a whole?"	Count by non-unit fractions and understand non-unit fractions as a count of x number of unit fractions		
	Partition objects or shapes into equal-sized parts, expressing the size of each part as a unit fraction			

Standard C.2. Generalizing Behaviors of Operations across Number Domains

C.2.a. Building understandings of the operations through contextualized situations
C.2.b. Fostering connections among the operations
C.2.c. Extending understandings of the operations to multi digit whole numbers
C.2.d. Extending understandings of the operations to all rational numbers

Providing opportunities for children to solve a variety of problems using their own strategies supports them in making sense of what the operations are and how they work. This standard focuses on how understanding of the operations develops and is extended. EMSs understand how children's informal ways of approaching problem solving situations relate to what adults understand as the more formally defined operations of addition, subtraction, multiplication, and division. They also understand the ways children's intuitive problem solving strategies can be leveraged to illuminate relationships and connections among the operations, and how these relationships and connections are extended across the number system, including multi-digit whole numbers, fractions, and decimals. Deep knowledge of the operations is critically important for later learning, with research showing that first-graders' knowledge of arithmetic is predictive of their ability to operate with fractions in middle school.[28]

Elementary Mathematics Specialists (EMSs) understand the algebraic underpinnings of computation across the number system, including how fundamental properties (e.g., commutativity) of number and operations are used naturally by children in their strategies for solving problems. Rather than isolating the operations and domains of number from one another, EMSs work to ensure that instruction emphasizes connections and coherence between whole numbers, fractions, and decimals. In doing so, EMSs create opportunities for learners to develop, articulate, and justify generalizations about how the operations work, and how these generalizations relate to a range of computational strategies and help to define a student's sense of number.

C.2.a. Building understandings of the operations through contextualized situations

Children enter school with a great deal of intuitive and informal knowledge of mathematics that can be built upon to make meaning of what the operations are and how they work. Without being told or shown how to solve problems, children as young as preschoolers will draw on their knowledge of real or imagined story contexts to generate solution methods that model the actions and relationships between quantities within the problem. Similarly, older children's work with operations may be inspired by playing games or constructing mathematical models that involve some form of data collection.[29] Collectively, these intuitive ways of reasoning about, representing, and solving problems allow children to mathematize

[28] Bailey et al., 2014.
[29] McClain et al., 2000; Sembiring et al, 2008

(i.e., decontextualize by translating from contextualized situations to mathematics) and construct viable solutions to a wide range of problems, recontextualizing their work to ensure strategies and solutions address the problem posed. Accordingly, the "operations of addition, subtraction, multiplication, and division can be defined in terms of these intuitive problem solving processes, and symbolic procedures can be developed as extensions of them."[30]

Developing robust, interrelated understandings of the operations involves making sense of a wide range of problem structures. For example, young children can begin to construct ideas about addition and subtraction by posing and solving problems that describe joining and separating situations with an unknown result. Young children can also begin to develop foundational meanings of multiplication and division by solving problems involving equal groups. Over time, children extend their views of addition and subtraction by solving problems that do not contain explicit actions, such as comparison or part-part-whole situations, or problems that are not easily solved by directly modeling a problem's sequence of actions (e.g., start-unknown situations). Children also learn to build upon their understandings of equal groups situations to make meaning of multiplication and division situations involving arrays, area, and multiplicative comparisons, with multiplicative comparison paving the way to understanding problems involving rates, ratios, and scaling at the middle school level and beyond. During elementary school, work with operations is extended to multi-digit whole numbers, fractions, and decimals, and includes multi-step problems involving more than one operation.

Contextualizing and Decontextualizing*.* *Students develop deep understandings of mathematics as move between the real world and related mathematics representations (i.e., model with mathematics). Given a real world situation, students can be asked to pose and solve problems using varied mathematics, or decontextualize the situation. Once they have a model that answers their question, which may require iteration, they recontextualize as they return the real world to verify their model is effective. Students may also enter this cycle when given a decontextualized situation and they create a context that connects to the real world.*

EMSs should understand the range of problem structures for addition, subtraction, multiplication, and division; how children (rather than adults) are likely to make sense of and represent these situations; the relative difficulty of modeling the actions and/or relationships of different problem structures; and potentially productive (or problematic) ways various problem structures can be incorporated into instructional sequences and instructional materials. EMSs understand that the contextualizing, decontextualizing, and recontextualizing of expressions and equations is an important part of making sense of the quantities and relationships represented by symbolic notations. EMSs recognize that when students construct their own strategies for solving

[30] Carpenter et al., 2015, p. 4.

problems, they develop an intuitive understandings of the properties of operations and the relationships among operations. Relatedly, building understandings of the operations through contextual situations can also provide an entry point into "reading and writing the world with mathematics" to interrogate social and community issues and inequities (e.g., from school-community issues of distributing snacks fairly and sharing school supplies to broader world issues of protecting salmon and accessing clean water).[31]

C.2.b. Fostering connections among the operations

Given opportunities to build from their intuitive strategies and to communicate with others about their ideas, children's natural approaches to solving problems will become increasingly abstract over time. For example, a kindergartener might initially solve a multiplication story about 3 packages of gum with 6 pieces of gum in each package by representing three packages, each of which contains 6 individual cubes, and counting all the cubes (by ones) to equal 18.[32] Later, a child might skip count by 6 three times, or derive the solution by knowing that 3 packages with 5 pieces of gum would equal 15, and one more piece for each package is 3 more, so 15 + 3 equals 18. Providing students with opportunities to wrestle with the similarities, differences, connections between, and varied representations of these approaches helps to build understandings of the relationships between and among the operations and develop skills in selecting the solution strategy best for them and the particular problem.

Representing and Connecting*.*
As students are provided opportunities to engage in making sense of and solving problems then recording their thinking in ways that are natural to them, they come to see connections both within and across contextual, verbal, physical, visual, and symbolic representations (e.g., models and drawings of 3 packages each with 6 pieces of gum with connected equations) as well as the similarities and differences among the operations (i.e., 6 + 6 + 6 = 3 × 6).

As the meanings of the operations develop, children's solution strategies and ways of representing their ideas become increasingly abstract. Initially, many children will use fingers, cubes, or drawings to represent quantities as composed of individual units, and then count (by ones) to operate on the quantities they have created. In this way counting both provides access to and unites the operations. For example, as children begin to count some quantities as composed units themselves, they might solve a story problem asking about the number of cars that will be needed to take 30 children on a field trip if 5 children can fit in each car by (a) finding the number of times 5 can be added to reach 30 by skip counting, or (b) the number of times 5 can be subtracted from 30 by repeated subtraction. In this case, the richness of children's varied approaches supports them in grappling with relationships between division, addition, and subtraction.

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[31] Arnold et al., 2021; Bartell et al., 2022; Chao & Jones, 2016; Gutstein, 2006; Koestler et al., 2022.
[32] Carpenter et al., 1993; Celedón-Pattichis & Ramirez, 2012; Turner & Celedón-Pattichis, 2011.

EMSs understand and support others in understanding the patterns of development in student-generated strategies, how students' solution paths are influenced by and related to different problem structures, and the connecting threads between concrete, semi-concrete, early deriving strategies, and those that are more abstract. In their work with students, EMSs support these connections by creating classroom environments in which students explain the details of their mathematical ideas and engage with their classmates' reasoning by drawing attention to similarities, differences, and connections. As they support students in noticing these patterns, they draw attention to the importance of selecting a strategy that makes sense for the problem and to them, recognizing that the most efficient and accurate strategy for a student may or may not be the standard algorithm. EMSs also understand how the properties of operations (e.g., commutative, associative, distributive) often underpin students' invented approaches, and the different ways students and teachers can represent varied solution paths. For example, a child might derive a solution for a join, change unknown story problem (7 + __ = 12) by recalling that 7 and 3 more equals 10, and 2 more equals 12, so 3 more and 2 more equals 5 (implicitly drawing on the associative property of addition, see figure 4). EMSs understand the importance of students communicating and justifying their use of the properties of the operations in varied ways. Students might use pictures, gestures, story contexts, or natural language to communicate the generalizations they are noticing about the operations.[33] For example, they might use two stacks of cubes to demonstrate the commutative property of addition, by "switching them" the placement of two stacks to show that the sum remains the same even if the order of the two addends is changed.

7 + __ = 12 (7 + 3) + 2 = 12 7 + (3 + 2) = 12 3 + 2 = 5

Figure 4

Illustration of a Child's Informal Use of the Associative Property of Addition within a Mental Solution Strategy

C.2.c. Extending understandings of the operations to multi-digit whole numbers

A rich foundation of reasoning and problem solving experiences supports children as they extend their ideas of how the operations behave as the number system expands to include multi-digit whole numbers. Counting (by ones) allows young children to solve problems involving multi-digit numbers before they understand the relationship between tens and ones or work with units of ten. Multiplication and measurement (also known as quotative) division problems involving groups of 10 are especially powerful as they invite children to create groups of ten from collections of individual units or to find the total created by iterating

[33] Schifter & Russell, 2022.

groups of 10. Children's strategies for solving these problems progress to reveal an increasing grasp of operating on 10 as a unit and of the relationships between tens and the base ten number system.[34] As children continue to solve more sophisticated problems, their expanding understanding of the base ten system allows them to solve multi-digit problems in ways that are closely related to single-digit strategies such as make ten. Recalling the child's single-digit strategy from above, a related strategy for solving a multi-digit join change unknown problem (70 + __ = 128) might be that 70 and 30 more equals 100, and 28 more equals 128, then 30 more and 28 more equals 58 more, so 58.

C.2.d. Extending understandings of the operations to all rational numbers

Young children can also begin to develop understandings of fractional amounts through solving equal sharing story contexts in which the leftover amounts can be partitioned and acted on in different ways depending on the context.[35] In solving these problems, children create informal representations of fractional amounts that can serve as the basis for developing formal understandings of fraction terms and symbols. For example, in attempting to solve a problem such as *If 4 friends want to share 6 sandwiches so that each person gets the same amount, how much of the sandwiches should each person get?*, children will often draw sandwiches and partition the leftovers into parts that represent halves or fourths before they can formally name the fractional amounts created through their solutions (see Standard C.1.c).

Once children can connect their concrete and semi-concrete representations of fractional amounts with the corresponding abstract terms and symbols, they can begin to make sense of whole numbers as composed of unit fractions ($2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$), and of unit fractions as the building blocks for other fractions ($\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$). Similar to previous work involving groups of 10, multiplication and measurement division problems involving groups of same-sized unit fractions are especially powerful in providing opportunities for children to iterate unit fractions and decompose wholes into unit fractions and to reason about these relationships.[36] For example, to find the number of days a guinea pig could be fed $\frac{1}{4}$ of a carrot each day if there were 6 carrots, a child represented each carrot as composed as 4 fourths, and found the total number of fourths within their drawing to determine the number of days the guinea pig could be fed (see figure 5). Because the strategies children develop for solving problems can involve drawing, skip counting, adding, and subtracting fractional amounts, children's approaches to solving what adults might recognize as multiplication and division problems connect operating on fractions, and can precede formal work with addition and subtraction (e.g., $\frac{2}{4} + \frac{1}{4} =$ __).

[34] Bray & Blais, 2017.
[35] Empson & Levi, 2011; Siegler et al., 2010.
[36] Empson et al., 2020.

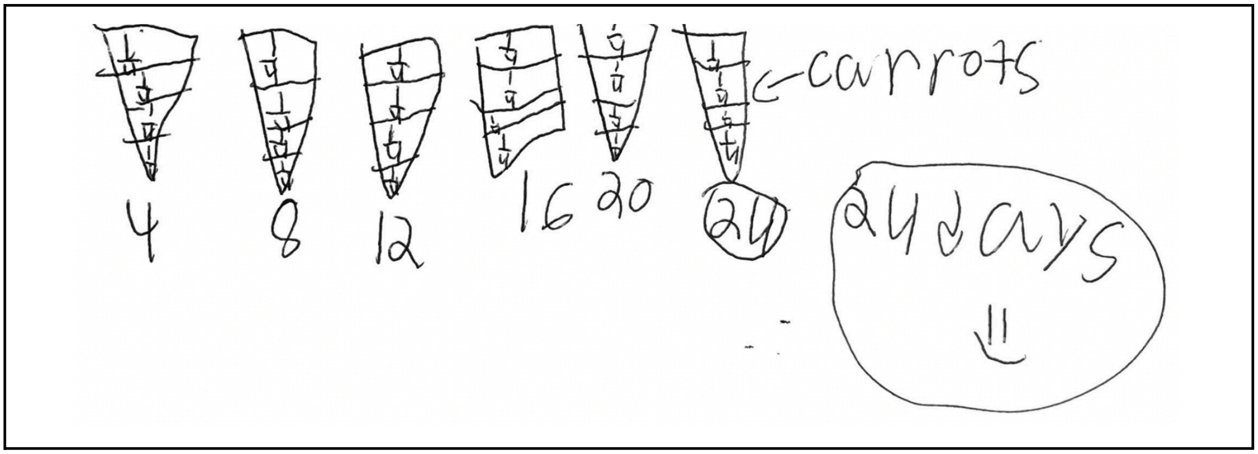

Figure 5

A Child's Solution for Finding How Many Days 6 Carrots Will Last if a Guinea Pig is Fed $\frac{1}{4}$ of a Carrot Each Day

EMSs understand the connections and relationships between children's strategies for operating on whole numbers and the ways that children's work within earlier number domains can provide a foundation for work with multi-digit whole numbers, fractions, and decimals. EMSs attend to the central role of both formal and informal representations in allowing children to both establish and extend their understandings of the operations and a variety of invented and standard algorithms across number domains. EMSs also understand how to build from children's understandings of fractions as quantities to make sense of fractions (including decimal fractions) as quotients, operators, and ratios. Importantly, EMSs know that students having a thorough understanding of PK-8 mathematics includes: robust interconnections of conceptual understanding of numbers and operations; flexibility and fluency with procedures; and the mathematical agency to select the best strategy for the given context.[37]

[37] NCTM, 2023c.

Figure 6

Mathematical Storyline for Operations across Domains

PK-Grade 1	Grades 2-3	Grades 4-5	Grades 6-7	Grades 8-9
Understand the meaning of and relationship between addition (put together, add to) and subtraction (take apart, take from, compare)	Understand the meaning of and relationship between multiplication and division (equal-sized groups, arrays, area models)		Understand the meaning of and relationship between operations with rational numbers	Understand scientific notation and use operations to solve problems where decimal and scientific notation are used
Extend understandings of counting to quantifying the number of objects that have been combined or what is left are some are taken	Extend understandings of and models for addition and subtraction to fractions	Extend understandings of and models for whole number multiplication and division to operations involving fractions and decimals	Extend understandings of and models for operations with fractions to all rational numbers	
Apply strategies based on place value (e.g., "making tens") and properties of operations	Apply intentionally selected and increasingly refined strategies for multi-digit addition and subtraction	Apply intentionally selected and increasingly refined strategies for multi-digit multiplication and division	Apply intentionally selected and increasingly refined strategies for rational number operations	Apply intentionally selected and increasingly refined strategies for solving linear equations and systems of linear questions
Solve problems involving equal groups to build understandings of grouping and connections between the operations	Solve problems to develop fluency with addition and subtraction within 20 and multiplication and division within 100	Solve problems to develop fluency with - multi-digit multiplication of whole numbers, and - fraction and decimal addition and subtraction.	Solve problems to develop fluency with - multi-digit division, and - decimal multiplication and division.	

Standard C.3. Recognizing, Extending, and Generalizing Mathematical and Everyday Patterns

C.3.a. Recognizing, extending, and making generalizations about repeating and growing patterns

C.3.b. Representing functions with visual models and contextual situations

Algebra is a useful tool for connecting mathematics and the real world as well as for sense making across all areas of mathematics. This standard focuses on the study of patterns and functions, and representing and modeling mathematical and everyday or imaginary contexts.[38] Supporting learning in this domain engages learners in the activities of noticing, extending, conjecturing, proving, and generalizing patterns, functions, and change. Developing these algebraic habits of mind is critical for future mathematics learning.[39]

Elementary Mathematics Specialists (EMSs) create opportunities for learners to engage in algebraic thinking by asking purposeful questions and the use of intentional task design. They know that with exposure over time to algebraic habits of mind such as seeking and identifying patterns, making connections, justifying reasoning, etc., learners begin asking “I wonder” and “what if” questions prompting student-directed exploration of relationships and eventually, making generalizations that were historically teacher-led. As students’ experiences across the grades engage them in developing a deep understanding of mathematics, these experiences simultaneously strengthen their mathematics identity and this strengthened identity, in turn, encourages them to explore mathematics deeply.

C.3.a. Recognizing, extending, and making generalizations about repeating and growing patterns

Patterns are found across mathematical domains as well as across the curriculum. They exist in number relationships such as the sequence of number names, the powers of ten represented by digits in the base ten number system, and proportional relationships that can be seen by skip counting by multiples of a number or across numbers such as 3s, 6s, 12s, etc. Patterns also show up within measurement contexts (e.g., the repeatable sequences of activity in daily life such as schedules, the cyclical nature of the calendar and seasons) and measurements themselves (e.g., the direct relationship between measures of attributes and the size of the unit as the attribute being measured changes, the inverse relationship between measures of attributes and the size of the unit as the unit changes). Students can explore patterns in the natural and constructed world or within scientific phenomena. For example, natural patterns include constant change in plant growth or snow melt; fractals in snowflakes and tree branches; spirals in pine cones and pineapples; reflective symmetries in animals and leaves; and rotational symmetry in many flowers and some animals. In construction, patterns

[38] Blanton, 2008; Kaput, 2008.
[39] Borriello et al., 2023; Zippert et al., 2020.

are used for both aesthetic and engineering purposes to make a design more beautiful, safer, or stronger.[40]

Explaining and Justifying.
Engaging in the process of exploring, developing and testing conjectures, and deciding what is going on in a situation, supports the ability to explain both "what" is happening, "why" it is happening, and the extent to which it will always happen. For example, as students explore factors of related multiples such as factors of 24 and 6. They notice all the factors of 6 are also factors of 24. This can launch an exploration of other related multiples and their factors to test the conjecture with more evidence and eventually justify why this is happening through use of a model or context.

EMSs understand that patterns can take multiple forms (e.g., growing and repeating patterns, symbolic and visual patterns), model various kinds of change (e.g., constant, linear, nonlinear, periodic), and can be represented in multiple connected ways (i.e., words, tables, symbols, graphs). They understand that when dealing with a function or rule, an input (*x*-value) corresponds to exactly one output (*y*-value) and that some functions have "rules" and some do not (i.e., some can be represented with an equation and some tell a more complicated story, such as melting ice or running a race). EMSs know how to support learners with seeing and describing change or covariance, specifically, how change in one variable (e.g., number of children, an independent variable) results in a predictable change in a related variable (e.g., number of ears, a dependent variable). EMS professionals understand that moving toward generalization requires opportunities for learners to: (1) *explore* a mathematical situation; (2) develop a *conjecture;* (3) *test* the conjecture to determine if it is true or false; (4) if the conjecture is not true, *revise* and test again; and (5) *generalize*–if the conjecture is true and justified with evidence.[41]

In addition, EMSs understand the characteristics of "simple" patterns that represent proportional relationships and the characteristics of more complex patterns (e.g., non-zero start value or non-proportional linear, nonlinear). They recognize the natural progression of describing relationships with words before expressing them with symbols. They understand the tendency of noticing recursive patterns before attending to the relationships between corresponding values. EMSs use their knowledge to develop intentional instructional moves that build upon recursive pattern observations in support of moving toward explicit rules that are possible when attending to correspondence. They use their "algebra eyes and ears"[42] to influence task selection, design, and implementation with the intent of developing algebraic thinking over time.

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[40] Arnold et al., 2021.
[41] Carpenter et al., 2003.
[42] Blanton & Kaput, 2003, p. 73.

C.3.b. Representing functions with visual models and contextual situations

Critical characteristics of functions show up in a variety of representations and features within each representation (i.e., contexts, words, tables, symbols, graphs) suggest functional growth and change that is linear or nonlinear (e.g., quadratic, exponential). Relatedly, contextualizing and decontextualizing patterns involves moving between abstract representations (i.e., symbols, graphs, tables) and potentially more relatable representations (i.e., visual and physical models, real world contexts) and vice versa. Taking time to translate between mathematics and the real world and look for connections within and across representations supports a deepened understanding of: (a) the function itself, (b) its defining characteristics, and (c) ways those characteristics show up in different representations.

Early work with functions often includes hands-on experiences such as using concrete objects or manipulatives. For example, consider the visual model using toothpicks to form a pattern with squares in a row (i.e., arrangement 1 has 1 square, arrangement 2 has 2 squares, arrangement 3 has 3 squares). One student wrote numeric expressions to communicate the growth and change in the toothpick squares as one square was added (see figure 7). Specifically, the student saw the pattern as adding 3 toothpicks each time after the initial 4 toothpicks–translating *across* representations from physical or visual to symbolic. Another student saw the pattern in a slightly different way, with 3 toothpicks for every square with one more to close the final square. This student recognized the equivalence across the two expressions stating, “3 + 1 at the end is the same as adding 4 at the beginning”–translating *within* representations from symbolic to symbolic.

Contextualizing and Decontextualizing. *When presented with an abstract representation such as an equation, table, or graph, contextualizing the situation can support sense making. For example, reading 15 pages of a book each day after beginning on page 3 represents the equation 3 + 15x. If considering a graph or table, the start value is not the origin (0, 0; or the beginning of the book) rather it is at 3 (0, 3) with a constant increase of 15 pages a day (creating a line or a change in y of 15). The habit of examining each representation for where the 3 and 15 can be seen, supports development of the ability to understand the relationships among different functions (e.g., 3 + 15x; 3 + 18x, 12 + 15x).*

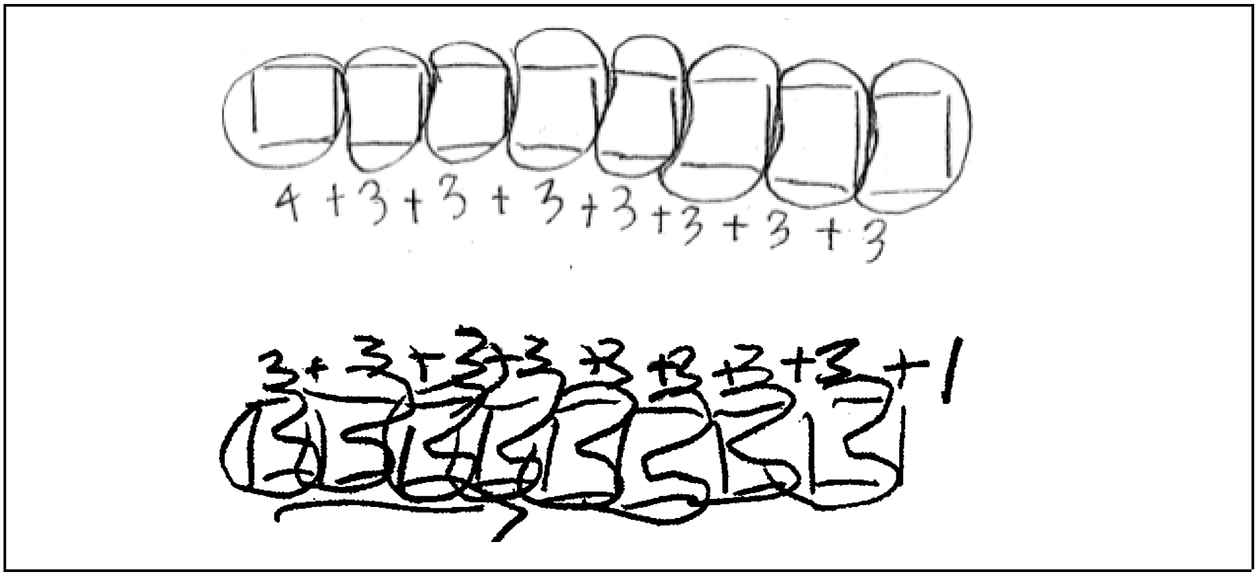


Figure 7
Students' Strategies for Seeing and Counting Toothpicks

Importantly, EMS professionals understand that representations are more than an end product of an investigation, rather they are tools for thinking and communicating about a given situation. EMSs support students' access to algebraic thinking by providing students time to make observations, explore, and connect a wide variety of informal and formal representations. EMSs understand the importance of drawing attention to relationships among constants, variables, and diverse representations as well as more generally noticing *what* is growing and *how* it is growing as this supports generalizations about situations. Figure 8 provides an example of what it might look like to intentionally build from observations about what is growing in the toothpick squares pattern toward how it is growing more generally. Because all the counting strategies focus on the same growing pattern, notation for each arrangement (e.g., loops and numeric expressions used to count the toothpicks in smaller, specific arrangements), and because they are consistent with the observations they can be generalized to algebraic expressions with meaning connected to their representations. Finally, while these methods generate different algebraic expressions (i.e., $4 + (n-1)(3)$, $3n + 1$, $(n)(4) - (n - 1)(1)$, $2n + (n + 1)$), those expressions are equivalent.

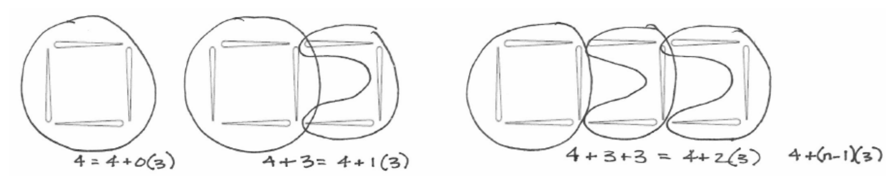
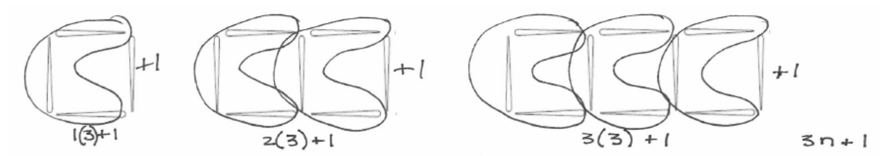
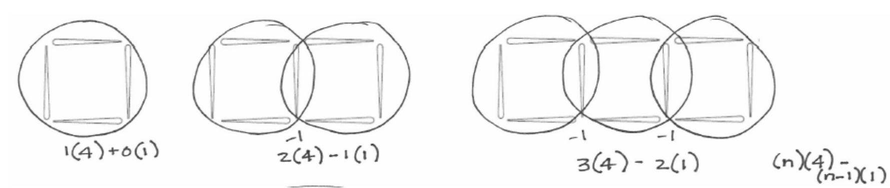
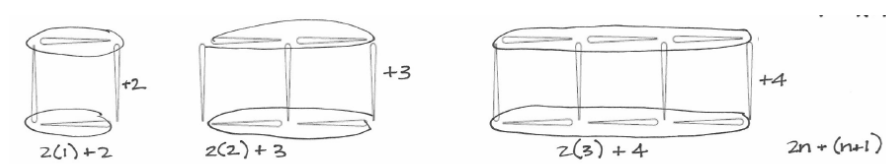
Observations about the Toothpick Squares	**Visual, Numeric, and Algebraic Representations**
● Starts with 4 toothpicks. ● Adds 3 toothpicks for each new square.	
● There are 3 toothpicks for every square in a C-shape. ● There is always 1 extra toothpick to the right.	
● There are 4 toothpicks for every square. ● Toothpicks of adjacent squares are double-counted. There is one less double counted toothpick than the number of squares.	
● The number of toothpicks on the top and bottom match the number of squares. ● The number of vertical toothpicks is one more than the number of squares.	

Figure 8

Building from Observations about the Toothpick Squares Pattern to Various Representations

Figure 9

Mathematical Storyline for Recognizing, Extending, and Generalizing Patterns

PK-Grade 1	Grades 2-3	Grades 4-5	Grades 6-7	Grades 8-9
Recognize, duplicate, and extend simple repeating and growing patterns	Recognize, describe, generate, and extend repeating and growing patterns	Recognize and describe structures of repeating and growing patterns (e.g., constant, variable)	Make generalizations about patterns and functions	
Duplicate and create patterns using shapes, sounds, actions, and words	Create and translate across pattern representations (i.e., verbal description, visual or story context, numeric expressions and equations)	Create and translate within and across pattern representations (i.e., verbal description, visual or story context, numeric and algebraic expressions and equations, tables, graphs)	Solve everyday and mathematical situations using numeric and algebraic expressions and equations	Translate among representations and describe how aspects of various functions are reflected in different representations
		Create rules using words or symbols (i.e., expressions, equations) to describe relationships	Make meaning of various explicit and recursive equations used to define a pattern	Understand a function as a rule assigning each input to exactly one output
		Compare numeric expressions to determine equivalence	Compare algebraic expressions to determine equivalence	Define, evaluate, and compare functions

Standard C.4. Exploring, Examining, and Enumerating Space

C.4.a. Recognizing, naming, describing, and comparing shapes

C.4.b. Composing, decomposing, and understanding space

C.4.c. Understanding spatial relationships and spatial structuring

C.4.d. Enumerating Space through Geometric Measurement

This standard focuses on geometric knowledge, spatial sense, and related applications, including measurement. These concepts are related to number and operation sense and connect to deeply understanding and describing the physical environment. Elementary Mathematics Specialists (EMSs) understand that the study of geometry provides students with opportunities to: understand geometric representations, reason logically, provide justifications, build connections among ideas, develop spatial and location skills, and visualize objects from different viewpoints.[43] These geometric understandings are essential in and of themselves, and also for their role in learning in other mathematics domains.

Research shows that spatial thinking is an important predictor of achievement in the STEM disciplines —science, technology, engineering, and mathematics, and that early spatial abilities impact later mathematics learning .[44] Elementary Mathematics Specialists (EMSs) understand that learners' in- and out-of-school experiences with geometric ideas influence their levels of geometric reasoning and spatial sense. They know how to select and sequence instructional opportunities to support movement through van Hiele's five levels of geometric thought: Level 1: Visual Level, Level 2: Descriptive Level, Level 3: Informal Deduction Level, Level 4: Formal Deduction, and Level 5: Mathematical Rigor.[45]

Notably, a focus on spatial reasoning provides multiple entry points and equitable access to mathematics. Geometric work can support culturally relevant connections as students look for and make use of their spatial reasoning to understand their world and to participate in mathematics activities that contribute to their environments and communities both in and outside of school (e.g., determining where the library bus can park to serve more families, designing dog pens for a new animal shelter).[46]

C.4.a. Recognizing, naming, describing, and comparing shapes

Young children enter school with many intuitive and experiential ideas about the shapes they have encountered in their world. They can sort and compare shapes or parts of shapes. For example a Kindergarten student might say, "the sailboat in the picture and the wedge of

[43] Jones, 2012.
[44] Mix, 2019; Verdine et al., 2017; Wai et al., 2009.
[45] van Hiele, 1999.
[46] Arnold et al., 2021.

cheese are both the same shape." Through experiences that allow students to compare and contrast shapes, students begin to move from sorting and classifying shapes by any attribute (such as color and size) to using attributes of shape such as number of sides or angles. Students begin to use the critical attributes of shape and relationships among shapes to develop definitions that allow them to create and make generalizations about shape categories. They can provide justifications to statements such as "all squares are rectangles, but not all rectangles are squares."

Explaining and Justifying.
It is through many experiences observing, acting upon, and describing features of shape and space that students discover important relationships among shapes (van Hiele Levels 1 and 2). Over time, students move from describing to developing informal deductive reasoning as they recognize the orientation of a shape does not change the features of shape or see that all quadrilaterals are 4-sided, it is just that some have more precise names based on angle measure or side length (van Hiele Level 3).

Two- and three-dimensional shapes can be recognized, named, described, sorted, compared, and ordered. Initially, shapes are recognized and named informally characterized by how they are seen and used in the real world (e.g., "a sailboat and wedge of cheese are both the same shape"). Later shapes can be more deeply analyzed in terms of characteristics of the overall (whole) shape as well as the component parts, defining attributes, and properties.

EMSs understand that students develop the language of geometry through experiences that allow them to observe, touch, sort, and create a variety of shapes. They also recognize the multifaceted nature of how children communicate their spatial ideas.[47] Precision of geometric language develops as students describe and compare shapes. Support for multilingual students that includes the use of visuals, rehearsals, and language cognates should be embedded in activities.[48] EMSs understand that the visualization of shapes in two and three dimensions requires that students view shapes from different viewpoints and orientations. EMSs recognize that it is important for students to work with a variety of shapes so that they do not develop restricted views of shapes that must later be unlearned (e.g., all triangles look like equilateral triangles and sit on a base). EMSs recognize that students work with different aspects of shapes at different grade levels and work with increasingly more complex shapes.

C.4.b. Composing, decomposing, and understanding space

At first, young students compose with different shapes to create pictures (e.g., a square and a triangle can make a house) or cover pictures made up of different shapes (sometimes by trial and error) by matching smaller shapes within an existing larger shape (e.g., tangrams).

[47] Hinestroza, 2022; Moschkovich, 2013
[48] Zwiers et al., 2017.

Students then begin to use defining attributes of shapes to compose new shapes, such as putting together two triangles with same side lengths or two right triangles to make a square. Students can discuss the parts and totals of the shapes they compose and decompose and use a new composed shape as a unit to make other shapes. Students' development of part-whole relationships plays a critical role as they decompose and recompose space. Ideas about congruence and similarity are developed as students manipulate and compare shapes. Key geometric ideas of equivalence include understanding that any two objects of the same size that occupy the same space and have different shapes are considered equivalent.

Students learn about two and three dimensional shapes by composing and decomposing them. Through these transformations they learn about the components and properties of shape. Their ability to visualize, describe, and transform geometric regions develops alongside their ability to recognize and use iteration of measurement units, the construction of patterns, the decomposition and composition of numbers, including fractions.

EMSs understand that the compositions and decompositions of regions are important for solving a wide variety of area problems. For example, the area of a triangle can be determined by composing a rectangle made of two of those triangles as shown in figure 10a. Also, knowledge of finding the area of rectangles and triangles can be extended to find the area of any two dimensional shape, essentially composing, decomposing, and rearranging through rotations, reflections, and translations to find the known shape (see figure 10b for an example of how this thinking is extended to find the area of trapezoids).

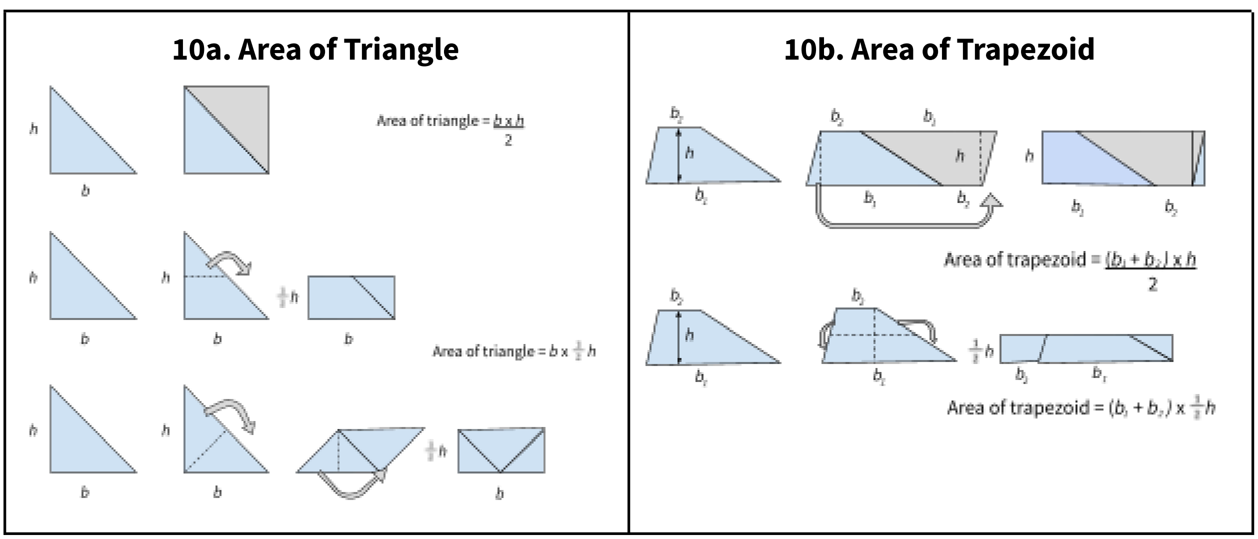


Figure 10
Three Ways of Thinking about the Area of a Triangle and Why the Formula Works (10a)
Strategies for Finding the Area of a Triangle Extended to Finding the Area of a Trapezoid (10b)

EMSs support this area of work by encouraging students to analyze and make use of structure. EMSs know how to develop experiences that allow students to engage in the creating, composing, and decomposing of units and higher order units. They understand that students move from pre-composer level to piece assembler to picture-maker to shape composers.[49] EMSs know that observing students as they compose and decompose shapes offers important information about how a student is thinking about space. Students might begin by using trial and error to cover a shape composed of individual shapes. Later on, students are able to visualize blank spaces as composed of individual shapes and can mentally manipulate shapes. EMSs are able to draw upon students' ideas about the composition and decomposition of shape as students develop strategies and their own formulas for finding the area and volume of a variety of shapes. EMSs understand that students' strategies are later extended to shapes that may have side lengths or units that are fractional or decimal amounts.

C.4.c. Understanding spatial relationships and spatial structuring

Young students develop spatial reasoning through puzzle play, board games, or building with blocks or geometric shapes.[50] Early work with composing and decomposing lays the foundation for spatial structuring. Positional words and phrases such as above, below, next to, and inside of are used to describe spatial relations first informally, and then later on in more formal ways. Additional visualization work can occur through experiences with picture books, dot images, or shape images. Spatial structuring involves the mental operation of constructing an organized form of an object or set of objects. Students can use structures such as arrays to move from seeing discrete objects to a set of squares in rows and columns. These structures are used first with two dimensional shapes and then the arrays can be layered to measure the volume of the three dimensional shapes (see figure 11).

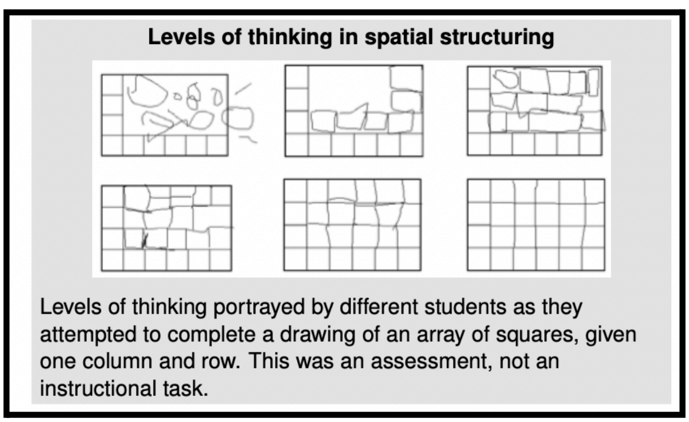


Figure 11
The Typical Progression of Spatial Structuring for an Array

[49] Clements et al., 2004, geometry trajectory https://www.learningtrajectories.org/math/learning-trajectories.
[50] Parks, 2015; Wager & Parkes, 2014.

Spatial reasoning plays a fundamental role in our everyday lives and also plays a vital role across all mathematical domains. Researchers have found connections between spatial reasoning and the understanding of number, measurement, and problem solving.[51] The National Research Council[52] has identified three aspects of spatial reasoning: (a) concepts of space; (b) tools of representation; and (c) processes of reasoning. Understanding concepts of space includes understanding relative distance or size as well as aspects related to continuity and dimensionality.

EMSs recognize the important role of viable explanations and justifications as students solve problems related to spatial relationships and structures. They attend to and support students' cultural competencies, including the use of gestures and developing language as strategies for describing spatial relationships. EMSs understand and leverage the strong relationship between number and spatial reasoning by encouraging the use of varied representations (e.g., concrete objects, drawings, area formulas) and the manipulation of shapes and structures to explain and justify, for example, properties of the operations, such as using an array to develop a justification for the commutative property. EMSs are aware that the segmenting of space can present students with instances in which there is a need to use fractional units. They recognize the importance of students having opportunities to segment space in units that may be expressed as whole numbers, fractions, or decimals. These experiences support understandings related to number line diagrams (e.g., recognizing fractional amounts as a point on a number line) and coordinate axes (e.g., graphing points on the coordinate plane).

C.4.d. Enumerating space through geometric measurement

Children bring many informal measurement experiences from everyday life to their more formal school-based encounters. They use informal words such as *tall or big* to describe many dimensions of objects. Young students begin this work by using informal methods of measuring and comparing objects. At first, they have to identify what attribute of an object they are going to measure, and then decide what tools and strategies they will use to do so. The use of direct comparisons, that is, comparing an attribute of two objects without measurement (e.g., two students stand back to back to see who is taller) is number free and allows for a focus on the attribute that is being measured.[53] The use of indirect comparisons (i.e., using a third object to compare two objects) occurs as students look for and make use of measurement benchmarks such as a doorway or a pencil. During the elementary grades, qualitative perceptions progress to more quantitative descriptions. Later on, these quantitative descriptions can become numeric comparisons between an object and a specific unit.[54]

Measurement of space is a contextual application of number sense and spatial reasoning.

[51] Drefs & D'Amour, 2014.
[52] NRC, 2006.
[53] Clements & Battista, 1986.
[54] Schifter et al., 2017.

Geometry and measurement provide important contexts that develop, deepen, and allow for application of whole number and fraction knowledge and related understandings. Many specific aspects of an object can be measured such as dimension (e.g., length, diameter), area, volume, angle size, etc. Measuring any attribute requires an understanding of that attribute as continuous that can be subdivided into smaller iterations or units, rather than as a count of discrete objects. This continuous measurement unit is in contrast to cardinality, which involves a discrete attribute.

For example, an area can be found using an array (a collection of discrete units arranged in rows and columns) or an area model (a continuous measure, multiplying the side lengths). Geometric measurement spans across three dimensions of space (i.e., 1D, 2D, 3D), includes both regular and irregular shapes, and attends to the measurable aspects of those shapes (i.e., length, area, volume, angle). In later grades, students apply their understanding about features of shape, transformation, and equivalence to make generalizations about measurement formulas and their use (see figure 12).

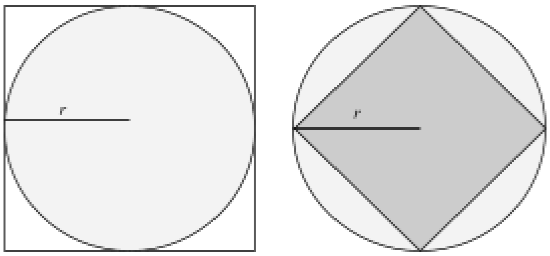
EMSs recognize the role of reasoning and problem solving in geometric measurement, and the importance of regular problem solving opportunities that foster geometric thinking. They understand that the use of measurement formulas without conceptual understanding negatively impacts students' progress toward developing deep understandings of geometric measurement. EMSs support the use of non-standard and standard units through explorations that lead to the need for more precise measurements (i.e., measuring all the space, no gaps or overlaps). They design and implement tasks that allow for the internalization of units and measurement processes. EMSs understand that geometric measurement involves a continuous property of space (i.e., length, area, volume) and that the iteration and counting of units requires the use of spatial structuring. They understand the value of students engaging in three dimensional problems that involve geometric relationships, such as tasks that involve nets and thus provide contexts linking two dimensional and three dimensional relationships.[55]

EMSs consider the complexity of ideas related to units and unit iteration. They foster understanding of these ideas for teachers and students by developing measurement experiences that illuminate the following: the selection of a unit must match the attribute of the object being measured; the iterated unit must be a constant size (and orientation if a non-square or non-cubical unit); the need to measure the entire space of focus without gaps or overlaps; and an understanding that the smaller the unit is, the more units are needed and the more precise the measure. However, there always will be some degree of error.

[55] Driscoll, 2007.

12a. Strategies for Approximating Area of a Circle

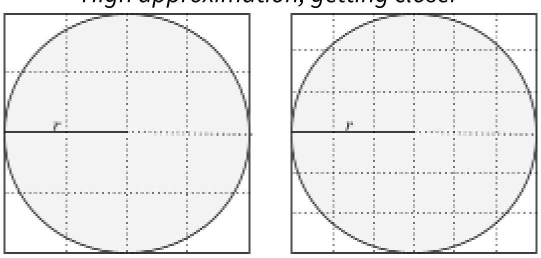
Low approximation



$Area_{large\ square} = 4r^2$, $Area_{small\ square} = 2r^2$

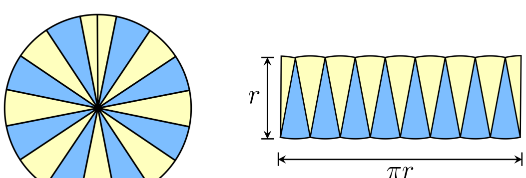
So, $\sim Area_{circle} = 3r^2$

High approximation, getting closer



$$\sim Area_{circle} = \left(\frac{r}{2} \cdot \frac{r}{2}\right) \cdot 13 = \frac{13}{4} \cdot r^2$$

$$\sim Area_{circle} = \left(\frac{r}{3} \cdot \frac{r}{3}\right) \cdot 28 = \frac{29}{9} \cdot r^2$$



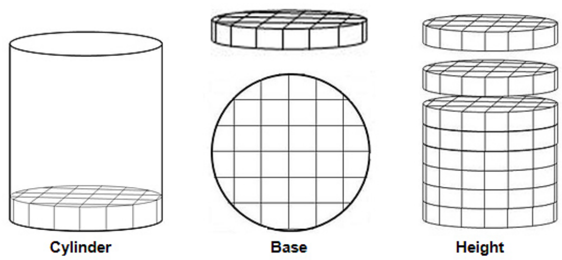
https://en.wikipedia.org/wiki/Area_of_a_circle#/media/File:CircleArea.svg

$Circumference = 2\pi r$,

$Half\ Circumference = \pi r$

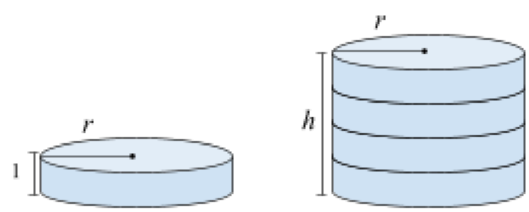
$Area_{circle} = \pi r \cdot r = \pi r^2$

12b. Strategies for Approximating Volume of a Cylinder



https://www.texasgateway.org/resource/determining-volume-cones-and-cylinders

Volume is layers of cubic units, height is number of layers



Volume is layers of circular discs, height is number of layers

$Volume_{cylinder} = \pi r^2 h$

Figure 9

Varied Ways of Approximating the Area of a Circle (12a), Extended to the Volume of a Cylinder (12b)

Figure 13

Mathematical Storyline for Exploring and Enumerating Space

PK-Grade 1	Grades 2-3	Grades 4-5	Grades 6-7	Grades 8-9
Communicate about objects in terms of shape name, size, and orientation	Compare, and classify shapes by number of sides and angles, connect with definitions	Analyze and classify shapes based on properties of their lines, angles, symmetries	Reason about relationships in scale drawings	Use ideas about distance, angle, congruence, and similarity to describe and analyze behavior of 2D figures under translations, rotations, reflections and dilations
Compose and decompose geometric shapes	Build and analyze 2D and 3D shapes to build foundations for area and volume	Compose and decompose geometric solids	Decompose shapes to find and justify areas and volume by rearranging or removing pieces and relating the shapes to rectangles and rectangular solids	
Compare and measure length by iterating same-size units (e.g., connecting cubes, popsicle sticks, paper strips) in connection with using standard measurement tools	Understand and measure - length (i.e., dimensions, perimeter) using standard units and tools - area using same-size units that cover space (e.g., tiles, playing cards)	Understand and measure - area using standard units and tools - angles (e.g., wedges in folded circle, pattern blocks) - volume using same-size units (e.g., cubes)	Work with 2D and 3D shapes to solve problems involving area, surface area (e.g., nets), and volume	Understand and apply the Pythagorean Theorem

Standard C.5. Investigating Questions and Interrogating Data through Statistical Problem Solving

C.5.a. Collecting, considering, organizing, and representing data

C.5.b. Selecting and using appropriate statistical methods to analyze data

C.5.c. Interpreting data: developing inferences and evaluating predictions

C.5.d. Understanding basic concepts of probability

Statistics differs from other mathematical sciences because of "the focus on variability in the data, the importance of context associated with the data, and the questioning of data… [making statistics] particularly relevant for all fields of study."[56] Statistical problem solving involves investigating questions and interrogating data in a non-linear process that includes: formulating questions; collecting or considering the data; analyzing the data; and interpreting the results (as shown in figure 14). Worthwhile investigations depend on question formulation that in addition to having a question that anticipates variability, there must be clear variables and intentions, and the question must be answerable with primary or secondary data. Regardless of the data source, addressing each component of the process is essential, as is constantly returning to the question and potentially identifying new investigative questions.

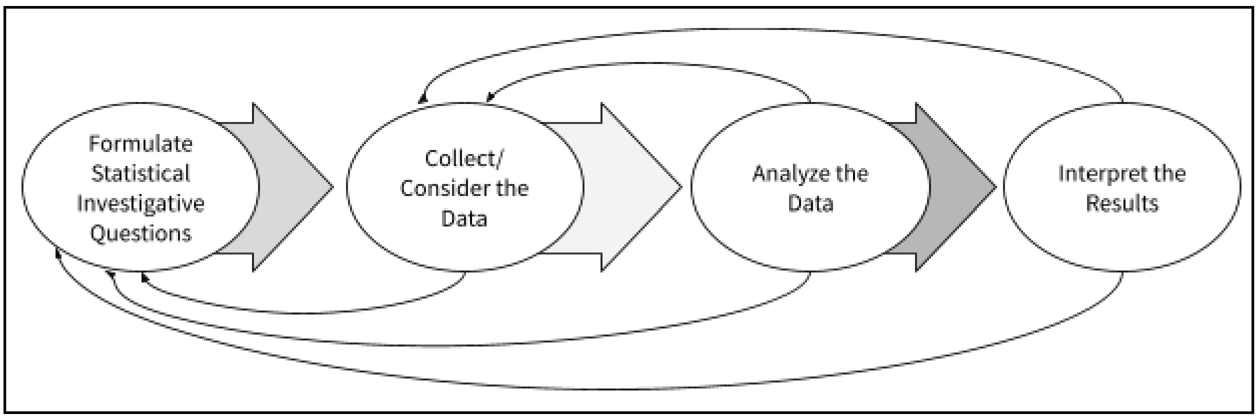


Figure 14
The Statistical Problem Solving Process[57]

Probability examines and eventually defines the likelihood of a situation. The key to early probabilistic thinking is understanding that some events are more likely or less likely to occur than others. In later grades, probability provides contexts for making data-based predictions.

Elementary Mathematics Specialists (EMSs) know that children at very young ages have

[56] Bargagliotti et al., 2020, p. 8.
[57] Bargagliotti et al., 2020, p. 13.

innate notions of variability and probability. EMSs understand that they can capitalize on these notions by exploring data and probability questions that emerge naturally from the classroom and school environment (e.g., fair distribution of snacks or school supplies, likelihood of going to recess after lunch, or that more teeth were lost by older students at the school). They recognize the importance of question formulation and provide opportunities for students to pose questions that they can answer by collecting or considering existing well-defined data sets. EMSs recognize that engagement in these activities builds upon student experiences with counting, sorting, comparing, ordering, and operating on whole numbers, fractions, and decimals as well as provides interesting contexts to apply their mathematical understandings and practice skills. Similar to the modeling with mathematics cycle, statistical investigations may include interrogation of culturally-relevant, real-world contexts (e.g., overcrowded classrooms, water shortages, homelessness, climate change), and these applied experiences not only develop data reasoning skills but also demonstrate mathematics is a useful tool for answering questions and critiquing the world in which they live. When engaging in the statistical problem solving process, EMSs support students and teachers in recognizing the value of curiosity and skepticism as they consider the questions to ask, the data to collect, the foci for analysis, the evidence influencing the interpretation of results, and the best way to report their findings. Finally, EMSs recognize the importance of early work with these data science concepts is what prepares students for the data-based decision-making necessary in their lives both in and out of school.

C.5.a. Collecting, considering, organizing, and representing data

Early work with data focuses on answering statistics-related questions from data that students collect or data that has been collected by others. These data may be collected through direct observation, simple surveys, or measurements from simple experiments. As students engage in collecting, considering, and organizing data, representation opportunities become available. While there are conventions to eventually learn, students benefit from repeated experiences of coming up with their own approaches to collecting, organizing, and representing data, later comparing these invented approaches with standard approaches to understand the benefits and drawbacks of each. Later, students should be allowed to think about and represent multiple variables at a time, and how to represent and analyze data using a variety of representations supported by appropriate technological tools (e.g., Common Online Data Analysis Platform [CODAP], Polypad, IES's Create a Graph). Statistical problem solving centers on recognizing and understanding variability, and then representing that variability in ways to support analysis and ultimately interpretation.

Initial data investigations include categorical data with sorted physical objects (e.g., shoes, stuffed animals) arranged to form physical bar or circle graphs that can be translated into picture graphs, bar graphs, and circle graphs. Numerical data may be represented using line plots and frequency tables and then in later elementary grades and beyond, expanded to include histograms, stem-and-leaf plots, line graphs, scatterplots, and box plots. Once students have experience creating and analyzing various types of data displays, technology can be a useful tool for creating interactive dynamic representations that support deepened analysis and discussion.

EMSs understand how to design learning progressions from simple, single-variable pre-collected data, to well-defined groups of interest from whom to collect data, to more complex, multi-variable contexts with data that may include errors or missing values. They also recognize there are many opportunities for student voice and choice when it comes to data investigations. EMSs design data contexts that 1) connect to students' lives, giving them voice as they bring their expertise and interest to statistical problem solving; and 2) empower students by giving them choice about how to collect, organize, and represent data. In either case, EMSs know they can better engage all students when they build from what students know and what makes sense to them.

C.5.b. Selecting and using appropriate statistical methods to analyze data

Data analysis can begin with a completed data representation or once students have formulated questions and collected and organized data. When given the opportunity to notice and wonder, students very naturally observe larger and smaller amounts, how the data seem to clump together, and other stories the data may be telling. From there, they can make informal decisions related to range, frequencies, and "what's typical" within a data set. As students more formally analyze data, the most common descriptive statistical methods used relate to measures of center (i.e., mean, median, and mode) and variability (i.e., range, standard deviation) of the data.

Students will often recognize how data collection and representation of data are both similar and different. They should informally discuss range as they note both the largest and smallest amounts (e.g., monthly rainfall accumulation), or most and least often chosen selections within a set of categorical data (e.g., weather pattern choices from among sunny, cloudy, rainy, snowy, or other). Consider the following investigation of typical household sizes for students in Mrs. Lopez's class as a way to build intuition about measures of center and range without actually applying a statistical method or computational procedure. The following question was posed: *"How many people, including yourself, live in the*

Noticing and Using Mathematical Structure. *Long before students formally explore measures of center and variability they notice attributes of data sets as they collect and organize data. For example, when examining the "typical" number of goals scored in soccer games, students noticed that in most games either 1 or 3 goals were scored, but in one game a non-typical 8 goals were scored. As they progress through the grades, students formalize their observations using standard data displays and statistical methods or computations and recognize the connections between these structures and earlier intuitions.*

household you are living in now?"[58] First, nine students represented their family size using connecting cubes. Then they arranged the stacks of snap cubes in the increasing order of their family sizes (see figure 15a). As the connecting cube data were represented, students could recognize that family sizes varied and there are some family sizes that show up more often. Given the chance to notice and wonder about these data provides opportunities to discuss the range of family size; the most frequent or modal number of family members; and good estimates of both the median and mean number of family members. To develop conceptual understanding of mean as "equal sharing," students can explore different approaches to equalizing the cube column lengths to find the average or mean family size (see figure 15b) and consider how the results would change if two of the nine students had three more family members. These observations about the distribution of family sizes can provide the opportunity for teachers to formally name and define terms as well as consider which measure of center is the most useful in describing the typical family size in Mrs. Lopez's class. They can also engage students in thinking about and discussing how changes in a data set impact the distribution. Middle grades students will build upon these understandings as they learn how to determine the mean, median, range, and interquartile range and use these measures to compare data sets.

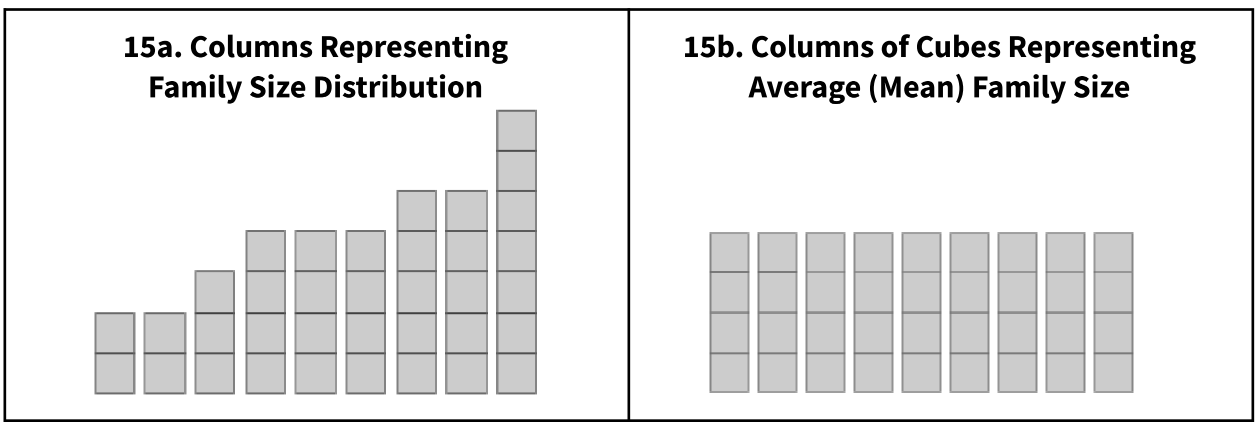


Figure 15

Columns of Cubes Ordered by Number of Family Members (15a). Equalized Length Columns of Cubes Representing Average Family Size (15b).

EMSs recognize the importance of providing students with opportunities to create and examine representations of distributions to get a sense for the overall shape of (e.g., clumps, gaps, symmetry) and variability in the data. They also understand the benefit of slowing down the analysis to informally notice and name characteristics of the data that can later be connected to more formal analysis through statistical methods or computations. Finally, EMSs support the development of strong data sense in preparation for the work in later grades that focuses on the interrelatedness and representativeness of the measures of center and variability (i.e., interquartile range, mean absolute deviation).

[58] Adapted from: Bargagliotti et al., 2020, p. 28-29.

C.5.c. Interpreting data: developing inferences and evaluating predictions

Data analysis includes representing and describing key features of data distributions to support data-based decision-making that may involve making inferences or predictions. For example, an analysis of student attendance data collected over a week could support predictions about attendance for the following month while taking into consideration trends related to illness or upcoming events that may affect the data.

From the time that younger students begin categorizing and organizing data, they should be encouraged to both acknowledge and discuss differences that might explain the variability between groups. With more experience, for example, students can make statements about the results from a survey of third graders about their field trip preferences even when results vary from one class to the next.

Data are all around us. EMSs know that students need to regularly consider, frame, and respond to questions involving data, be able to understand and critically think about the use of data, and finally develop inferences or evaluate predictions based on data. They recognize the importance of statistical literacy in navigating today's world and work with teachers to consider the multitude of mathematical and statistical learning opportunities that data analysis provides.

Contextualizing and Decontextualizing. *Moving between the real world and the mathematical world is central to the statistical problem solving process–with formulating questions, collecting, representing, and considering the data and what is happening in context, decontextualizing when creating models and analyzing the data, and recontextualizing when interpreting the results.*

C.5.d. Understanding basic concepts of probability

Early and informal understandings related to probability are related to whether a particular event is likely to occur. Prior knowledge and related understandings of numbers and in particular fractions, will assist students in understanding that the probability of an event occurring is the fraction of the time that event will occur theoretically (e.g., if you need to spin a 4 on an equally divided 1 through 4 spinner, the probability of the spinner landing on 4 would be $\frac{1}{4}$, one chance in 4 trials). Students engage in probabilistic contexts by conducting and analyzing the results of experiments.

EMSs support teachers as they plan and facilitate instruction on probability at the elementary school level, which would typically focus on understandings related to likely and unlikely events as well as representing and understanding the results of simple experiments (e.g., games, surveys) and perhaps making and testing predictions. Such probability-related opportunities extend work with data analysis and fractions.

Figure 16

Mathematical Storyline for Statistical Problem Solving with Data and Chance

PK-Grade 1	Grades 2-3	Grades 4-5	Grades 6-7	Grades 8-9
Formulate questions about data	Formulate investigative questions and interrogate data, including the differences between data sets (e.g., how many data points, distribution of quantities)	Frame, discuss, and resolve questions about data, including questions related to variability within contexts involving measurement (e.g., daily low and high temperatures)	Recognize and solve problems related to the variability of a data set	Identify the sources of variability within a data set, and draw inferences based on data analysis.
Collect, consider, organize, and represent data informally as well as in conventional ways (e.g., real graphs, tables, picture graphs)	Represent categorical and numerical data using picture and bar graphs and scaled picture and bar graphs	Represent data using line plots, scaled line plots, coordinate graphs and line graphs	Represent data by using dot plots, histograms, and circle graphs	Select and use appropriate representations of data, including box plots and scatterplots
Compare quantities with up to 3 categories	Describe relationships in data sets, answering “how many more,” “how many less,” and least and greatest (i.e., variability) questions	Informally describe data sets, determining “what is typical” about the data, and the differences between the data, building intuitions about measures of center (i.e., mean, median, mode) and variability (i.e., range)	Analyze and interpret data sets by determining measures of center and variability (i.e., interquartile range, mean absolute deviation)	Use and interpret measures of center and variability from random samples and draw informal comparative inferences about the sampling
			Interpret how measures of center, range, and interquartile range are influenced by changes in data	
		Discuss and describe events as likely or unlikely	Understand that the likelihood of an event can be represented using a number from 0-1	Determine the probability of compound events using organized lists, tree diagrams, and area models

Mathematics Pedagogy: Knowledge, Skills, and Dispositions

Elementary Mathematics Specialists (EMSs) support the classroom and grade-, school-, or district-wide use of ambitious and equitable mathematics instruction. They serve to enhance the planning, teaching, learning, and assessing of mathematics, and provide critical direct and indirect support for improved student mathematical understanding and identity development. In these efforts, EMSs have robust mathematics pedagogical knowledge and practices, with this section describing what EMSs understand and implement with PK-6 students as well as support others (e.g., general education teachers, special education teachers, teacher candidates, paraeducators) in doing so. It is important to note that the standards within the pedagogy domain build upon one another, from knowing students to using that information in support of the planning, implementing, and assessing cycle. Also, while the emphases of the standards—knowing, planning, teaching, and assessing—are presented separately, they are inherently and necessarily interwoven when it comes to the pedagogical knowledge and practices for teaching mathematics. Further, in the varied roles through which EMSs support ambitious and equitable mathematics instruction, they draw heavily upon their mathematical knowledge for teaching, including the knowledge, skills, and dispositions described in the Mathematics Content Standards.

EMSs' varying roles in schools and districts, such as generalist or specialist teacher, interventionist, leader, or coach, hold differing responsibilities that may involve work with students, teachers, or a combination of both.[59] If working with students, these standards describe the pedagogy-focused knowledge, skills, and dispositions needed to support student learning. If working with teachers, the standards describe the pedagogical understandings, capabilities, and mindsets to support teachers in developing. Outlined in the Mathematics Leadership Standards are some of the specific approaches for supporting collegial learning with particular attention to the unique learning needs of adult learners.

While this section focuses on pedagogy in support of student learning, much of this pedagogy is also relevant in support of teacher learning. For example, EMSs know and leverage teachers' diverse backgrounds and mathematical competencies during professional learning. EMSs have asset-based views and focus on mathematical strengths in their work with teachers in order to foster teachers' confidence in mathematics. EMSs create a community of learners where teachers are encouraged to be curious, take risks, and ask questions, which cultivates teachers' sense of feeling supported and receptiveness. Further, teachers are encouraged to reason and make sense of their mathematical thinking, within a context that affirms and elevates multiple and varied strategies for solving problems. During interactions, EMSs position teachers as holders of expertise and assign competence through naming or publicly drawing attention to teachers' contributions. Over time, teachers learn to recognize, value, and rely on their own mathematics strengths as well as those of others. These are just a few examples of the pedagogy presented here that is also applicable to teachers as learners.

[59] Association of Mathematics Teacher Educators [AMTE] et al., 2022; Rigelman et al., 2024.

Standard P.1. Knowing Students to Foster Positive Mathematics Identity

P.1.a. Knowing students' diversities and strengths to develop positive mathematics identity

Standard P.2. Planning for Responsive Instruction and Making Curricular Decisions

P.2.a. Planning for instruction based on learners and meaningful content

P.2.b. Understanding and using curriculum effectively

Standard P.3. Implementing Ambitious and Equitable Mathematics Instruction

P.3.a. Fostering a mathematics learning community

P.3.b. Focusing on a deep understanding of mathematics

P.3.c. Leveraging multiple mathematical competencies

P.3.d. Affirming students' diversities and mathematics identities

Standard P.4. Assessing Student Understanding and Learning

P.4.a. Engaging in the formative assessment cycle

P.4.b. Using summative assessments formatively

Standard P.1. Knowing Students to Foster Positive Mathematics Identity

P.1.a. Knowing students' diversities and strengths to develop positive mathematics identity

In addition to Elementary Mathematics Specialists (EMSs) knowing and recognizing students' varied ways of thinking and reasoning about mathematics and engaging in the mathematical practices and processes[60] (see Mathematics Content Standards), these professionals know their students' diverse mathematical competencies and cultural and linguistic backgrounds in order to be responsive and leverage them during instruction. EMSs also know their students' mathematical strengths, so they can focus on and build upon them. Grounded in this knowledge, EMSs aim to develop positive mathematics identities of students, and support their colleagues with seeing the value of deeply knowing students in order to foster positive mathematics identity.

P.1.a. Knowing students' diversities and strengths to develop positive mathematics identity

In EMSs' work with students, they know and value each student as an individual and plan intentional learning experiences that account for and leverage the diversity of students' backgrounds and mathematical understandings.[61] This diversity includes but is not limited to developmental variations, neurodiversity,[62] disability, race/ethnicity, language, gender, sexual orientation, economic class, country of origin, culture, community, and interests. EMSs recognize that students' mathematics identities need to be fostered, and knowledge of learners is a critical step for doing so in order to build upon students' varying mathematical competencies as well as their community, culture, language, lived experiences, and interests. Positive mathematics identity allows students to see themselves as competent, confident, and capable knowers, doers, and sense makers of mathematics, which undergird strong agency as active participants in mathematics.[63]

EMSs believe, and support teachers in developing beliefs, that each and every student is capable of learning mathematics through ambitious and equitable instruction. Coupled with these beliefs, in their work with students EMSs view all as mathematically brilliant, while accessing and building upon students' prior knowledge and informal knowledge during instruction. Rather than emphasizing what students do not know or understand, EMSs have an asset-based lens to identify and leverage students' existing understandings. All students have mathematical strengths, and EMSs focus on and teach to these strengths. They do this by noticing, naming, and documenting these strengths, and using these understandings to

[60] National Council of Teachers of Mathematics [NCTM], 2020.
[61] Aguirre et al., 2024; Chval et al., 2021; Gutiérrez, 2018; NCTM, 2021; Seda & Brown, 2021; Tan et al., 2019.
[62] Lambert & Harriss, 2022.
[63] Aguirre et al., 2024; NCTM, 2020.

design instruction.[64] By affirming and building upon mathematical strengths, students' confidence in their mathematical abilities will flourish, developing their identity as mathematicians.

EMSs disrupt deficit-based thinking and views about mathematics learning and students, especially related to race/ethnicity, economic class, gender, culture, language, and ability. They also disrupt associated deficit-based practices such as labeling and sorting of students into ability-based groups and tracks based upon test scores.[65] Further, when communicating with students, families, teachers, school administrators, and others, EMSs practice care in their language so asset-based views of students are expressed. EMSs support teachers in similarly having asset-based views and approaches to students and their learning as well as challenging spaces of marginalization.

EMS as Grade-Level Mathematics Coach: In my role as grade-level mathematics coach, I was supporting my 5th grade team as they were unpacking and planning their data unit. During the planning, the team started discussing the different types of graphs they would pull from already-created instructional resources to help students analyze data. Instead, I suggested that we start the unit by collecting data about students and their interests. The team talked about the different things their students were interested in and how questions about those interests could be answered and resulting data represented. One teacher shared that her students are obsessed with soccer, and we decided we could create a line plot of the number of goals scored by players on a soccer team each season. Another teacher suggested that the students could do something in class on which we could collect data. They suggested that we give the students some scrap paper and have them draw as many smiley faces as they could in 30 seconds. This led to a conversation about all the things students could do for 30 seconds to collect additional data (e.g., determine the number of words read, how many times they can write their name, the number of paper balls thrown into a box). Excited about the ways they were opening up the learning, another teacher added that students could decide on a way to represent the data and then teachers could use this as a way to revisit the standard forms of graphs they know and introduce others. I was thrilled the teachers decided to shift their focus of this unit to include student voice and choice based on their own interests and questions. I used this as an opportunity to point out how these changes can help to foster their students' mathematics identities.

Relevant Indicators: C.5.a., P.1.a., P.2.a., P.3.d., L.2.a.

[64] Kobett & Karp, 2020.
[65] National Council of Supervisors of Mathematics [NCSM], 2020; NCSM and TODOS, 2016; NCTM, 2020, 2023a.

Standard P.2. Planning for Responsive Instruction and Making Curricular Decisions

P.2.a. Planning for instruction based on learners and meaningful content

P.2.b. Understanding and using curriculum effectively

In their work with students, Elementary Mathematics Specialists (EMSs) plan for instruction that centers on learners and provides opportunities for each and every student to learn meaningful, important, and relevant mathematics. They use their knowledge of students (P.1.a) to develop instruction that cultivates positive mathematics identity. EMSs make knowledgeable curricular decisions, using understandings of curriculum standards progressions grounded in coherently sequenced content within and across PK-6 grade levels. This is coupled with a focus on essential understandings. EMSs practice agency by drawing on their expertise, professional judgment, and situated understandings when navigating school contexts and conditions. Accordingly, they promote the use of curriculum[66] in ways that are responsive to students' mathematical variabilities and backgrounds. Similarly, EMSs support teachers in this purposeful planning, which could occur through co-planning and assisting teachers with judicious decision-making related to varied instructional resources.

P.2.a. Planning for instruction based on learners and meaningful content

In EMSs' work with students, they use knowledge of students, mathematics, and students' varied ways of thinking and reasoning about the subject to plan ambitious instruction that promotes each student's access and learning. The designed instruction provides equitable, responsive opportunities for all students to learn meaningful mathematics, specifically focusing on conceptual understanding, procedural fluency, problem solving, and reasoning, and the development of a productive mathematical disposition.[67] Importantly, developed lessons: account for and support students' varying mathematical competencies and multifaceted, intersectional identities (multilingualism, neurodiversity, disability, etc.); draw on students' funds of knowledge[68] (i.e., home, culture, language, and community-based knowledge and experiences); humanize mathematics by providing opportunities for students to see themselves represented in the subject in meaningful and positive ways; and cultivate positive, collaborative relationships in a learning community. Using their knowledge of students' mathematical strengths, EMSs build upon these when planning lessons, including supports, scaffolds, modifications, and extensions.[69]

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[66] The term "curriculum" references both the "what" (e.g., outcomes, standards, objectives, guidelines) and "how" (e.g., instructional resources, learning experiences, interactions) of teaching and learning. The term "instructional materials" is used to refer to both core materials and supplemental resources.
[67] NCTM, 2014, 2020; National Governors Association Center and Council of Chief State Schools Officers, 2010; National Research Council, 2001.
[68] González et al., 2006.
[69] Kobett & Karp, 2020.

In preparing mathematics learning opportunities for students, EMSs attend to a multitude of factors, including research on how children learn mathematics, mathematical goals and standards, students' strengths, students' learning needs, relevance to their particular learners and context, instructional task selection, and information from classroom-based formative assessments. Particularly important to the planning process is the establishment of mathematical goals to focus learning, including developing clear goals for the mathematics that students are to learn, situating these goals within research on how children learn mathematics, and using these goals to develop lessons and select cognitively-demanding tasks.[70] When planning, EMSs: anticipate students' likely array of responses when engaging with a task; determine questions to ask of students who use specific strategies; and consider how the different strategies are related to the mathematical learning goals, connected to one another, and prioritized during whole class instruction.[71] Part of the planning process is critical analysis of instructional tasks, particularly for cognitive demand and relevance for their particular learners and context as well as potential biases embedded in the task, which would necessitate modifications. Further, EMSs plan learning opportunities that value and elevate student voice and choice. For example, during lessons students have options related to: materials, such as manipulatives; knowledge expression, such as drawings, models, language, or other methods; exploration of questions that are of interest to them; and differing learning tasks. When considering this purposeful planning, EMSs regularly engage in supporting teachers in doing the same, which would include careful vetting of the myriad and readily available resources for teaching mathematics, especially those online.

P.2.b. Understanding and using curriculum effectively

EMSs select, use, adapt, and determine the effectiveness of mathematics curricula within and across PK-6 grade levels. They evaluate the alignment of district-supported instructional resources and classroom-based assessments with state standards and assessments, and recommend any needed modifications. EMSs understand the importance of curriculum standards based on coherent sequencing of mathematics content topics within and across PK-6 grade levels (i.e., vertical progressions), coupled with a focus on the big mathematical ideas. They understand that deep and well-connected mathematics learning demands coherent curriculum and associated student learning experiences, including language used, materials (e.g., manipulatives) shared, and strategies highlighted.[72] Additionally, EMSs support teachers in understanding these vertical progressions of curriculum standards and the related need for cohesive and consistent student learning experiences across grade levels and settings (e.g., classroom, intervention groups), with a focus on the essential understandings.

EMSs use and support the use of high-quality instructional materials with integrity of

[70] NCTM, 2014, 2020; Stein et al., 2000.
[71] Smith & Stein, 2018; Smith et al., 2020.
[72] Karp et al., 2020.

implementation that thoughtfully accommodates local conditions and contexts.[73] Implementing these materials with integrity means EMSs adapt the resources in ways that are responsive to students' mathematical strengths, needs, and backgrounds. EMSs use their professional agency and support teachers in being agentic when it comes to curricular decision-making.[74] Specifically, EMSs draw upon their expertise, professional judgment, and contextual knowledge when selecting and implementing instructional materials and navigating conditions such as curriculum mandates, pacing guidelines, and high-stakes standardized assessments.

EMS as School-Level Mathematics Leader: Last year, my principal asked me to be the mathematics lead for our building. This role requires me to attend division-wide meetings and chair the mathematics committee at our school. During our division-wide meeting last month, we looked at our new mathematics standards and examined the vertical alignment for rational number and operations. I brought this work back to the mathematics committee and suggested that we continue to look at the vertical alignment to ensure that educators working at all grade levels and across student populations know where our students are coming from, where they need to go, and ways our instructional materials do and do not yet support the progression. These conversations led us to consider developing a school-wide agreement where we will discuss the "rules that expire"[75] and the progression of models throughout the grades, which will help deepen our collective content knowledge and support our students with clear and consistent vocabulary, structures, and expectations. I am excited to continue to develop agreements and embed them into our school culture.

Relevant Indicators: C.1.c., C.2.d., P.2.b., L.1.b., L.4.a.

[73] NCTM, 2020.
[74] AMTE, 2022.
[75] Karp et al., 2014.

Standard P.3. Implementing Ambitious and Equitable Mathematics Instruction

P.3.a. Fostering a mathematics learning community

P.3.b. Focusing on a deep understanding of mathematics

P.3.c. Leveraging multiple mathematical competencies

P.3.d. Affirming students' diversities and mathematics identities

In their work with students, Elementary Mathematics Specialists (EMSs) implement ambitious mathematics teaching that is equity-centered and identity-affirming. They foster a mathematics learning community with shared authority and power, supporting all students' voices, thinking, and participation. EMSs follow through with well-crafted plans for lessons and exhibit flexibility when warranted by evidence of student thinking, without lowering the cognitive demand of instruction. Importantly, they understand and implement research-informed teaching practices, specifically those characterized as ambitious and equitable in the National Council of Teachers of Mathematics' *Catalyzing Change in Early Childhood and Elementary Mathematics.*[76] These include the eight *Mathematics Teaching Practices*[77] and five equity-based teaching practices[78]. When the following eight *Mathematics Teaching Practices* are implemented together, the interconnections support ambitious and equitable mathematics instruction:

- Establish mathematics goals to focus on learning
- Implement tasks that promote reasoning and problem solving
- Use and connect mathematical representations
- Facilitate meaningful mathematical discourse
- Pose purposeful questions
- Build procedural fluency from conceptual understanding
- Support productive struggle in learning mathematics
- Elicit and use evidence of student thinking

As EMSs intentionally and explicitly attend to equity and access during instruction, they also draw from the following five equity-based mathematics teaching practices:

- Going deep with mathematics
- Leveraging multiple mathematical competencies
- Affirming mathematics learners identities
- Challenging spaces of marginality
- Drawing on multiple resources of knowledge.

During lessons, EMSs focus on developing a deep understanding of mathematics, drawing from their knowing students (Standard P.1), planning (Standard P.2), and assessing (Standard

[76] NCTM, 2020.
[77] Huinker & Bill, 2017; NCTM, 2014.
[78] Aguirre et al., 2024.

P.4). EMSs support teachers in understanding and enacting this instruction. Support for teachers developing ambitious and equitable instruction could be provided via professional learning that includes participating in lesson study, analyzing classroom video and student work, and engaging in mathematical tasks, along with modeling instruction, co-planning, co-teaching, and observing instruction then providing feedback and engaging in thoughtful reflection.

P.3.a. Fostering a mathematics learning community

In EMSs' work with students, they foster an inclusive mathematics learning community that centers on and elevates all students' knowledge and experiences.[79] They challenge and disrupt spaces of marginality, as they understand and strive to ensure that all students' voices and ideas are a necessary condition for learning mathematics. EMSs establish classroom expectations for participation (which could be co-created with students), position all students as capable, monitor how students position one another and manage status issues as they arise, and press for the academic success of all learners. When students' social and emotional needs are met, including their feelings of competence, safety, and belonging within the mathematics classroom, they more readily exhibit curiosity, take risks, and ask questions. A safe environment supports students in understanding that productive struggle is part of the process of learning mathematics, allowing them to willingly grapple with emergent or partial ideas. Further, within a context of valuing students' voices and choices, EMSs provide options during lessons related to materials, knowledge expression, questions for exploration, and learning tasks.

In cultivating a mathematics learning community, EMSs understand the importance of sharing mathematical authority with students by positioning them as holders of expertise. EMSs encourage all students to reason and make sense of their mathematical thinking and value their multiple, varied strategies, grounded in a firm belief that all children are highly capable mathematical learners and doers.[80] As EMSs notice students' mathematical strengths, they assign competence through naming or publicly drawing attention to students' intellectual contributions when solving problems.[81] This elevated positioning of student contributions as competent raises the status of each and every learner. Over time, students learn to recognize, value, and rely on their own mathematical strengths as well as those of others when engaging in mathematical tasks.

EMSs foster the social nature of learning mathematics by ensuring active student participation and collaboration in a discourse-rich environment. They prompt for students' mathematical thinking by using purposeful questioning techniques that facilitate productive classroom discussions focused on and responsive to student reasoning and sense making.

[79] Aguirre et al., 2024; Kalinec-Craig & Robles, 2020; NCTM, 2020.
[80] Carpenter et al., 2015; NCTM, 2014, 2020.
[81] Cohen & Lotan, 2014; Featherstone et al., 2011; Johnson et al., 2022.

Questions should assess and advance students' understandings, with EMSs using students' statements and work to build shared mathematical understandings for the class. This discourse-rich environment provides opportunities to make competence explicit by highlighting and amplifying students' contributions and recognizing emergent ideas. Throughout classroom interactions, EMSs elicit students' mathematical ideas, and attend, interpret, and respond to students' thinking as it unfolds, which serve as a guide for instructional decisions and enacted teaching moves.[82] Further, student groupings during learning experiences are intentionally mixed to leverage a variety of strengths, where expectations are high for all students and they collectively learn from one another.

EMS as Generalist Teacher of Mathematics: I am in my sixth year of teaching 1st grade, and one thing I always promote in my classroom is an encouraging mathematics learning community. To ensure that all students see themselves as mathematicians, I make sure manipulatives are accessible to students at all times and in open containers so they can choose what they need when they need it. For example, to support students' understanding of base ten concepts I give choices of manipulatives, such as those that are groupable (e.g., linking cubes, bundles of popsicle sticks, counters and cups) and those that are pregrouped (e.g., base-ten blocks, ten-frame cards), so if needed students can continue to build groups of ten to solidify the concept that 1-ten is a unit made up of 10-ones. Further, as students work on the day's task, I monitor and confer with individuals and groups of students using assessing and advancing questions to see where they are and where I can move their thinking along. For example, when asked to represent the quantity 23, I ask assessing questions about the connection between the numeric and physical representations, pointing to the 2 and asking, "Where do you see this in your collection?" and then pointing to the 3 and asking the same question. I am listening for them to account for all 23 objects rather than 2 ones and 3 ones. For advancing questions I might ask, "Another student told me [pulling 2 ones aside] this is the 2. What do think of that?" or "Last year I had a student build 1-ten and 13-ones for 23. Does this also show 23 and how do you know?" Another way I love to build community is through discourse routines such as "I used to think… Now I think…" I like to highlight for the whole class what can be learned through making mistakes and revising thinking, listening to and learning from others, and justifying reasoning. To me, this is a great way for all my students to feel confident from the beginning of my mathematics lesson to the end.

Relevant Indicators: C.1.b., P.3.a., P.3.b., L.2.b.

P.3.b. Focusing on a deep understanding of mathematics

In their support of students' learning, EMSs develop clear, rigorous learning goals, situate these goals within research on how children learn mathematics, and facilitate

[82] Jacobs & Empson, 2016; Jacobs et al., 2010.

cognitively-demanding instruction focused on meaningful, important, and relevant mathematics.[83] Mathematics learning goals should guide instructional decisions, including the selection and implementation of cognitively-demanding tasks. EMSs use tasks that: engage students in problem solving, reasoning, and sense making; allow for multiple solution strategies and the use and connection of multiple representations (e.g., visual, physical, symbolic, contextual, verbal); support connections across mathematical concepts; and develop conceptual understanding as a foundation for procedural fluency. EMSs support students in analyzing, comparing, justifying, and proving their solution strategies, which often occurs through collaborative student discussion and debate. Generally, EMSs support students' learning of content through engagement in mathematical processes and practices (e.g., representing and connecting, explaining and justifying, contextualizing and decontextualizing, noticing and using mathematical structures), rather than memorization of tips, tricks (e.g., mnemonics without connections), and rules and practice of step-by-step procedures and skills. In addition, in developing students' understanding of mathematics, EMSs use the subject as a lens for understanding, critiquing, and creating change in their world.

As EMSs focus on developing a deep understanding of mathematics, they know the meaning of conceptual understanding and procedural fluency and recognize that student learning must begin with and build from a solid foundation of deep conceptual knowledge.[84] EMSs recognize that conceptual understanding (i.e., the comprehension and connection of mathematical concepts, operations, and relationships) precedes and is necessary for developing procedural fluency, and they structure student learning experiences accordingly. Further, EMSs know that procedural fluency includes much more than remembering facts or applying standard algorithms; this fluency is complex and involves applying procedures efficiently, flexibly, and accurately. EMSs' instruction focuses on students becoming skillful over time in procedural fluency as they engage in contextualized problem solving, reasoning, and sense making. When procedures are connected with the underlying concepts, students benefit through improved retention, the ability to apply procedures in different and unfamiliar problems and contexts, and more productive disposition.

EMSs use mathematics-specific tools, such as physical models and technological tools, to support students' problem solving and reasoning. EMSs recognize that students can use physical models to represent and readily adjust their thinking. They know physical tools and manipulatives also provide access to problems and support communication with others. Technological tools also have the potential to enhance mathematical understandings by amplifying the mathematics beyond what can be accomplished with physical materials.[85] Technology has the potential to support mathematical visualization, modeling, and sense making. EMSs use technology to pose inquiry-based problems and facilitate creation of

[83] NCTM, 2014, 2020.
[84] National Research Council, 2001; NCTM, 2014, 2020, 2023c.
[85] NCTM, 2023b.

conjectures and justification of generalizations about mathematical topics. Additionally, EMSs encourage the thoughtful use of various physical and technological tools and support both teachers and students in understanding the affordances of each.

P.3.c. Leveraging multiple mathematical competencies

In their work with students, EMSs support the mathematical thinking and contributions of students with varying mathematical understandings and levels of confidence. This support is provided through implementing cognitively-demanding tasks and structuring student collaborations and discussions that allow them to use their differing mathematical knowledge and experiences to solve problems together. Students think, reason, and solve problems in different ways, and EMSs intentionally highlight and affirm this variability. For example, EMSs use cognitively-demanding tasks that support student access through multiple entry points to a problem, and they provide scaffolds and prompts or extensions based upon students' needs. In this, students with varying understandings and confidence can engage and make contributions. The tasks also allow for varied and invented solution pathways and knowledge expression.[86] This knowledge expression could include use of drawings, models, language, gestures, or other methods. The leveraging of multiple mathematical competencies also includes EMSs accessing, connecting to, and building upon students' prior mathematical knowledge. Throughout these learning opportunities, EMSs emphasize students' mathematics strengths so their confidence in their mathematics abilities is fostered.

P.3.d. Affirming students' diversities and mathematics identities

EMSs facilitate student learning experiences that account for and leverage students' diversity and develop their positive mathematics identity, as described in P.1.a. and P.2.a.[87] In responsively teaching students, EMSs differentiate the content, process, or products without lowering the cognitive demand for students. Instructional tasks concurrently have high levels of cognitive demand and to the extent possible connect to children's questions, interests, and lives by building on family, community, and cultural funds of knowledge.[88] Further, EMSs use story contexts and participants that mirror their own students' identities, experiences, and values as well as those of others.[89] Learning experiences for multilingual students emphasize multiple modes of thinking and communication (e.g., direct modeling, drawing, writing, speaking, gesturing), aiming to concurrently develop students' mathematical understanding and academic language.[90] EMSs understand, value, and leverage varying student conceptions, centering on student thinking and validating knowledge and experiences, both those in-school and out-of-school.

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[86] Schoenfeld & the Teaching for Robust Understanding [TRU]Project, 2016.
[87] Aguirre et al., 2024; Chval et al., 2021; Gutiérrez, 2018; NCTM, 2020, 2021; Seda & Brown, 2021. .
[88] Civil, 2007; Drake et al., 2015; González et al., 2006.
[89] Yeh & Otis, 2019.
[90] Chval et al., 2021; Moschkovich, 2015; NCTM, 2022.

Too often students with disabilities have limited opportunities to take on the role of doers of mathematics and to see themselves or have others see them as mathematically competent.[91] EMSs recognize the critical importance of students with disabilities, as well as neurodiverse students, coming to a robust understanding of mathematics through opportunities to: engage with the subject using their own solution strategies and reasoning, communicate their ideas, respond to others' ideas, and see connections between ideas. As elaborated in the indicators above, in working with students with disabilities, EMSs intentionally cultivate inclusive learning spaces and account for, leverage, and highlight multiple forms of mathematical competencies and knowledge. EMSs collaborate with others to develop meaningful mathematical goals in Individualized Educational Plans (IEPs) and to design and implement ambitious and equitable instruction. EMSs serving in the role of interventionist are highly knowledgeable of mathematics education research that focuses on students with disabilities. They draw from this knowledge for multiple purposes, including understanding and using effective instructional interventions and the multiple layers of support for students in schools. They also use these research-based understandings to advocate for students' capabilities and rebut deficit notions, such as the widely circulating belief that students who struggle automatically need explicit instruction and cannot benefit from instruction focusing on problem solving, reasoning, and sense making.[92]

EMS as Mathematics Interventionist: I was hired as a mathematics interventionist to support students' unfinished learning and their growth toward grade-level mastery. Many of my students tell me that they dislike math and struggle to find confidence. The intervention program I was given uses a lot of worksheets that focus on basic facts, procedural skills, and timed activities. However, I know that students need a deep understanding of why mathematics works the way it does, and that worksheets and a focus on speed will not affirm the abilities and competencies of students who struggle with mathematics. Instead, I provide students with hands-on materials so students can see what is happening in story contexts. For example, right now my 2nd-grade group really loves using the number rack. We explore interesting problems that students represent and solve using the number rack and then share their thinking with the group. In other grades we might use technology tools, like virtual manipulatives as numbers get large and the physical manipulatives become unwieldy. I know this shift from worksheets to a focus on understanding and building on their strengths is the best way for my students to access grade-level mathematics content and build their confidence.

Relevant Indicators: C.2.a., P.2.a., P.3.a., P.3.d., L.1.a.

[91] Lambert, 2018; Tan et al., 2019; Yeh et al., 2020.
[92] Lambert, 2018.

Standard P.4. Assessing Student Understanding and Learning

P.4.a. Engaging in the formative assessment cycle
P.4.b. Using summative assessments formatively

Elementary Mathematics Specialists (EMSs) understand that the primary purpose of assessment is to inform planning and instruction. In their work with students, EMSs elicit and analyze children's mathematical thinking to inform classroom interactions and instructional decisions, and they support teachers in these capabilities. EMSs recognize that the ongoing, iterative classroom-based assessment process of data gathering, analysis and reflection, and responsive changes in instruction (i.e., formative assessment cycle) is central to the effective teaching of mathematics. They recognize that classroom instruction must involve continuous assessment of students' mathematical understandings and growth through observation, documentation, and other varied and appropriate assessment tools and strategies. Further, they embrace the assessment cycle as a valuable tool that ensures continuity in students' development and learning experiences. EMSs also recognize the value of using student summative assessment data to evaluate and modify instructional units and responsively support student learning.

P.4.a. Engaging in the formative assessment cycle

In their work with students, EMSs assess for student understanding and growth by selecting, modifying, or creating assessment opportunities to elicit information on students' progress toward rigorous mathematics learning goals. In using assessments, EMSs recognize and attend to the many valued mathematical learning outcomes, such as conceptual understanding, procedural fluency, reasoning, and problem solving, as well as to productive mathematical disposition. They use a variety of both formal and informal assessments, including pre-assessment and classroom-based formative assessment tools and strategies as well as summative assessments. In doing so, EMSs gather evidence of students' mathematical thinking in ways suitable for children and sensitive to their backgrounds, and that allow for varying knowledge demonstration. EMSs understand the purposes, strengths, and limitations of each assessment technique and strategy they implement. They use formative assessments (e.g., observations, questioning, conferring) throughout a lesson in order to monitor and build upon students' mathematical strengths and provide needed scaffolds, supports, modifications, and extensions.

EMSs analyze and reflect upon assessment information and data to support the learning of each student. They regularly collect information on students' progress and use information or data from informal (e.g., observations, interviews) and formal (e.g., assessment tasks, end-of-unit assessments) assessments to analyze progress of individual students, groups of students, and the class as a whole toward meeting rigorous mathematics learning goals. In order to mitigate bias, EMSs are acutely aware of the potential of their own culture and background affecting their judgment when analyzing student understanding. Using an asset-based view, analysis of assessment information and data should illuminate students' mathematical thinking and strengths, as well as learning progress and needs to support responsive teaching. This analysis should also be used to reflect upon the effectiveness of their planning and instruction, including how particular teaching moves supported or inhibited student understanding and subsequent instructional steps.

EMSs modify instruction based on analysis and reflection, grounding their instructional decisions in evidence of student thinking and reasoning. They assess progress toward rigorous mathematics learning goals and adjust instruction continually in ways that support and extend learning. These adjustments should improve learning for each and every student, whether during in-the-moment instruction or a subsequent lesson. They build upon students' mathematical strengths and productive beginnings, while addressing early, partial, or alternate conceptions (i.e., common errors, partial solutions, overgeneralizations).[93] In particular, formative assessment within a lesson allows responsiveness to learners, knowing when to support and scaffold or when to allow students to productively struggle. EMSs understand the connection between assessment and feedback and create a classroom culture that recognizes the importance of teacher-to-student, student-to-teacher, and student-to-student feedback.[94] These opportunities may be written or verbal, with the intent of fully engaging students in the assessment process.

Further, EMSs recognize the value in not grading all assignments, such as cognitively-demanding instructional tasks.[95] They recognize the limitations of a traditional grading approach for classroom-based formative assessments. They explore ungrading techniques,[96] and competence-based approaches where the emphasis is feedback, so students can focus on understanding and growth rather than points and grades. In addition to students being able to grapple with tasks without apprehension of everything being graded, they should be encouraged to revisit and revise their solutions to tasks in order to develop a deeper understanding of the mathematical content.[97]

P.4.b. Using summative assessments formatively

EMSs use student summative assessment data (e.g., end-of-unit tests, benchmark assessments) to evaluate and modify instructional units and support student learning. They analyze and reflect upon assessment results and use this information to guide their subsequent design of instructional units, such as lesson planning around mathematical goals, instructional tasks, and tools. They also use this information to guide areas where some students may need an opportunity to re-engage with particular mathematics concepts.

[93] Fennell et al., 2024; Schoenfeld & the TRU Project, 2016.
[94] Fennell et al., 2024.
[95] NCTM, 2021.
[96] Blum, 2020.
[97] Safir & Dugan, 2021.

Summative assessments should be used to promote students' self-monitoring and self-regulation of their learning. EMSs use these assessments as a form of feedback to students on their learning, through: sharing results before or without scores, highlighting accurate and emerging understandings, providing actionable steps on what and how to improve responses, and offering opportunities to revise responses.

EMSs modify summative assessments in ways that reflect knowledge of their students, such as altering story problem contexts and providing choices about number sizes, just as they do instructionally. Further, for some students in which additional assessment information is not needed, EMSs revise summative assessments so that students do not respond to the full set of provided items. EMSs understand the limitations and potential biases of summative assessments and, as needed, make changes. Finally, EMSs strive to use and support others in using tools and strategies that are ethically grounded and developmentally, culturally, ability, and linguistically appropriate to document developmental progress and promote positive outcomes for all children.[98]

EMS as Grade-Level Mathematics Leader: In my role as a grade-level mathematics leader, I spend a lot of my time co-planning with my grade-level team. In our collaborative planning, it has taken some time for my colleagues to feel comfortable enough to ask me how to connect their planning to include more formative feedback following their classroom-based assessments. I knew I needed to pivot! To adjust our grade level planning time toward the use of feedback, I will use examples of student work as a way to engage my colleagues as they consider not just the importance of feedback, but decision-making relative to providing opportunities for: (1) student-to-student feedback, (2) when and how they may seek student-to-teacher feedback, and (3) discussing, based on the student work reviewed, how, when, and what they would provide for teacher-to-student feedback. This has been an important learning experience for all of us. As for me, I have come to recognize the importance of adapting my teacher leader responsibilities to address emerging needs of my colleagues. And, my principal liked what I have been able to do and is trying to find a way for me to work with other grade-level teams before or after school.

Relevant Indicators: P.4.b., L.2.b., L.4.a.

[98] National Association for the Education of Young Children, 2019; NCTM, 2020.

Mathematics Leadership: Knowledge, Skills, and Dispositions

The mathematics content, pedagogical, and leadership knowledge and skills of Elementary Mathematics Specialists (EMSs) regularly intersect as they engage in the various responsibilities of their positions as generalist or specialist teachers, interventionists, leaders, or coaches.[99] EMSs are formal and informal teacher leaders and fully engaged partners in a school's or district's mathematics program. They support movement toward a shared vision of high-quality mathematics instruction by: (1) advocating for structures that support students and educators; (2) leading efforts to advance implementation of ambitious and equitable mathematics instruction and assessment; (3) collaborating with building and district leaders to offer ongoing mathematics professional learning; and (4) ensuring positive collaborative relationships and reciprocal communication among staff, families, and community members.

EMSs effect change in school settings through their work with both children and adults. As they work with PK-6 students, EMSs employ the knowledge of content and pedagogy outlined in the previous sections. They apply the leadership knowledge and skills described in this section as they advocate on behalf of students to ensure access to important mathematics and relevant engaging teaching. Further, as EMSs work with adults (e.g., general education teachers, special education teachers, paraeducators, administrators, families, community members), they use their leadership knowledge and skills to guide actions related to developing: deepened understanding of mathematical content, strengthened students' and adults' identities, coherent instructional practices, and carefully aligned curricular decisions. Through their day-to-day presence and work alongside others, they are able to significantly influence the impact of high-quality mathematics programs.

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[99]Baker et al., 2021; Bitto, 2015; Fennell et al., 2013; Hjalmarson & Baker, 2020; McGatha & Rigelman, 2013; Rigelman et al., 2024.

Standard L.1. Advocating for Ambitious Instruction and Equitable Structures for Students and Teachers

L.1.a. Influencing policy and practice to ensure equitable access to meaningful and important mathematics

L.1.b. Collaborating with others to ensure high-quality, rigorous core instruction and appropriate interventions aligned horizontally and vertically

Standard L.2. Advancing Implementation of Ambitious and Equitable Mathematics Instruction and Assessment

L.2.a. Sustaining use of ambitious and equitable mathematics teaching practices

L.2.b. Promoting asset-focused use of formative and summative assessments and data

L.2.c. Using summative assessment data for mathematics program improvement

Standard L.3. Activating Continuous Mathematics Professional Learning and Program Improvement

L.3.a. Assuming responsibility for their own ongoing learning

L.3.b. Designing and facilitating educator professional learning

L.3.c. Advocating for structures in schools that support ongoing educator learning and consistency in practice

Standard L.4. Developing and Sustaining a Culture of Collaboration to Support Mathematics Teaching and Learning

L.4.a. Using knowledge of adult learners and learning to design professional learning

L.4.b. Cultivating relationships and sustaining collaboration with families and community members

Standard L.1. Advocating for Ambitious Instruction and Equitable Structures for Students and Teachers

L.1.a. Influencing policy and practice to ensure equitable access to meaningful and important mathematics

L.1.b. Collaborating with others to ensure high-quality, rigorous core instruction and appropriate interventions aligned horizontally and vertically

Elementary Mathematics Specialists (EMSs) understand educational policy, how it is established and implemented at the local, state, and national levels, and the roles of school leaders, boards of education, legislators, professional organizations, and invested others in its formulation. EMSs know that mathematics has historically been a mechanism for maintaining the status quo with selected affordances for some populations, and they advocate for mathematics as a tool for empowerment, workforce opportunities, and social mobility. EMSs use this knowledge to advocate for each and every student's mathematics learning needs and their identity and agency development.

EMSs recognize that students are often: marginalized based on culture, race/ethnicity, language, economic class, ability, gender identity, and sexual orientation; and denied meaningful mathematics based in part on biased perceptions, stereotypes, and actions, which broadens the opportunity gap. Understanding this, EMSs use their professional agency and collaborations to change perspectives, practices, and structures. Specifically, they support colleagues in reflecting on their beliefs, privileges, and biases; press for practices that support ambitious and equitable teaching and improved student learning opportunities; and advocate for structures that support high expectations for all learners, access to rigorous mathematics content, the use of high-quality instructional materials across the school, and carefully considered approaches to assessment. They serve in positions of influence within their school, district, community, and profession, functioning as change agents who have the abilities to challenge and dismantle biased beliefs, practices, and structures.

L.1.a. Influencing policy and practice to ensure equitable access to meaningful and important mathematics

EMSs evaluate educational structures and policies that affect students' equitable access to high-quality mathematics instruction and take actions to ensure that all students have opportunities to learn meaningful, important, and relevant mathematics. They do this in part by evaluating the alignment of mathematics standards, instructional materials, and formative and summative assessments, and then making recommendations for addressing learning needs without limiting learning opportunities for students. For example, they encourage learning acceleration rather than remediation for students who may benefit from additional support to access grade-level content.

EMSs speak and act with intentionality. They aim to disrupt problematic practices and structures that marginalize students by instead expanding their opportunities to learn mathematics. EMSs are committed to eradicating mathematics as a gatekeeper, and instead advance the study of the subject and connect to workforce opportunities. They collaborate with school-based professionals to develop evidence-based interventions for students who would benefit from more time and support as well as those who would benefit from more challenge. EMSs share information with colleagues within and beyond their context regarding how local, state, and national trends and policies may impact classroom practices and expectations for student learning. Aware of structures that limit students' opportunities to engage in rigorous mathematics, EMSs work to dismantle inequitable structures such as student tracking, ability grouping, repeating the same Individualized Educational Plan (IEP) goals year after year for a student, and labeling based on test scores or similar factors, as well as teacher tracking, which limits students' access to experienced teachers.[100] All in all, EMSs challenge policies and practices that limit students' opportunities to learn and foster continuity of the mathematics program across the school and district.

EMSs attend to the ways historical biases have limited student access and are aware of local and regional efforts that may politicize mathematics education. They advocate for the rights and needs of all students and collaborate with invested others to secure resources (e.g., financial support, human and material resources, ongoing mathematics professional learning opportunities, collaboration time) that support each and every student's learning. EMSs also advocate for mathematics education and the profession in contexts outside of the classroom, school, or district (e.g., instructional materials adoption committees, state or local standards development, teacher evaluation standards and processes). They communicate effectively, using a culturally responsive, strengths-based approach with targeted audiences (e.g., school- and district-based professionals, students, families, community members, teacher candidates, university partners).

L.1.b. Collaborating with others[101] to ensure high-quality, rigorous core instruction and appropriate interventions aligned horizontally and vertically

EMSs collaborate with colleagues to develop knowledge and skills focused on research-informed ambitious and equitable mathematics teaching practices. These practices cultivate deepened mathematical understanding (e.g., implementing tasks that promote reasoning and problem solving, building procedural fluency from a deep and flexible understanding of mathematics, differentiating with an eye on the Mathematical Storylines provided in the Content Knowledge section of these guidelines) in a supportive and responsive classroom environment (e.g., inviting a broad range of representations, honoring multiple strategies, positioning students as experts, using high-quality instructional materials

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[100] National Council of Teachers of Mathematics [NCTM], 2020.
[101] Others may include grade level team members, resource teachers, special educators, paraprofessionals, administrators, supervisors, preservice teachers, professors, etc.

consistently and responsively). When working with colleagues across the school, EMSs use research on mathematics teaching and learning to identify, plan for, and facilitate instruction that meets the needs of all students, while attending to the simultaneous goals of developing broad and deep mathematical understandings and positive mathematics identity for students and colleagues regardless of where mathematics instruction occurs.[102]

EMSs provide, or support others with, just-in-time scaffolds, modifications, or extensions at the classroom or school level. They are the "go-to" mathematics educators in the building seeking to develop coherence across students' mathematics learning experiences both horizontally and vertically. They encourage responsive use of instructional materials and humane approaches to instruction and assessment. Whether working in the context of a pull-out or push-in model for intervention, EMSs advocate for learning experiences that actively engage students as mathematical doers, knowers, and sense makers with grade level content. They may do this through direct work with students or through work with teachers or paraeducators who provide mathematics instruction. In this work, EMSs support others to develop skills related to eliciting learners' informal knowledge and existing understandings, supporting development of new knowledge, and using the information responsively to guide next steps instructionally.

EMS as Mathematics Interventionist: In my role as mathematics interventionist, I was supporting students in a 3rd grade classroom during a unit on fractions. While collaboratively planning, the teacher expressed concern about teaching fractions, confiding she never really understood fractions and does not know how to respond when students get stuck. When in the classroom, I noticed Mirabel struggling to place certain fractions on a number line. I invited the teacher to come alongside me and listen while I conferred with Mirabel about her work. I initially asked her to tell me what she was trying to figure out. I followed by asking, "Are there some numbers that are easier to place than others?" She told me $\frac{1}{2}$ and $\frac{1}{4}$ were easy. I followed up by asking how $\frac{3}{4}$ compared to the fractions she already placed. She looked puzzled, so I handed her a blank sheet of paper and said, "If this whole piece of paper is 1 whole, can you show me what part of the paper is $\frac{3}{4}$?" I wanted to know if it was the meaning of the fraction or the linear model causing her difficulty. She folded the paper in half and in half again, creating 4 equal parts. Next she said, "$\frac{3}{4}$ would be all but one part." I could then draw her attention back to the number line asking, "Based on that idea, how might you think about $\frac{3}{4}$ on the number line? Can you fold the line as you folded the paper?" I was intentionally modeling how I could ask questions to learn more about student understanding and then use their prior knowledge to extend to new learning. The teacher and I met later to discuss my approach of building on what Mirabel already knew and using those ideas as a scaffold for her access to this task. I wanted her to see that

[102] Nasir, 2002.

this move did not take away the thinking from Mirabel.

Relevant Indicators: C.1.c., P.3.b., P.3.c., L.1.b., L.2.a.

Standard L.2. Advancing Implementation of Ambitious and Equitable and Mathematics Instruction and Assessment

L.2.a. Sustaining use of ambitious and equitable mathematics teaching practices

L.2.b. Promoting asset-focused use of formative and summative assessments and data

L.2.c. Using summative assessment data for mathematics program improvement

Elementary Mathematics Specialists (EMSs) have a significant impact on both students and educators as they collaborate with educators (e.g., classroom teachers, interventionists, resource teachers, special educators, paraeducators) to plan, teach, assess, and help to coordinate services for students with identified learning needs in mathematics. Their collaboration deepens educators' awareness of the "look fors" in terms of trajectories of mathematical understanding as well as engagement in the practices and processes. EMSs are a critical resource in supporting school-wide implementation of ambitious and equitable mathematics teaching practices that affirm and build upon students' varying backgrounds and mathematics competencies and strengths, regardless of who is providing instruction.

EMSs are aware of and engage in policy decisions related to the use of core instructional and intervention materials, associated assessments, and other curricular recommendations. They recognize ways these resources align with ambitious and equitable mathematics teaching practices and support adaptations that draw out and build upon students' strengths. A critical element of EMSs' responsibilities is the ability to guide and support educators as they teach mathematics with the intent of supporting each and every student's deep mathematical understanding (e.g., content understandings, strategies), habitual engagement in mathematical practices and processes (e.g., contextualizing and decontextualizing, explaining and justifying, representing and connecting, noticing and using structure) and productive mathematical and learning dispositions (e.g., seeing math as useful and worthwhile, persevering through struggle).

L.2.a. Sustaining use of ambitious and equitable mathematics teaching practices

Through their collaboration with educators, whom may or may not have preparation in teaching mathematics, EMSs work to ensure responsive instruction and assessment that equitably engages each and every student in solving and discussing learning tasks that draw upon existing understandings and promote deep, well-connected mathematical understandings, reasoning, and problem solving. EMSs help educators make sense of mathematics content and practice standards in ways that support them in identifying key "look fors" in understanding and actions (e.g., realizing students may return to using direct modeling strategies as they move from work with whole numbers to decimals; noticing the difference between explanations that focus on "how" and those that extend to the "why"). EMSs establish and maintain a focus on ambitious and equitable teaching by working with educators one-on-one or in grade-level teams as they: anticipate student thinking; implement mathematical routines, tasks, and activities; and reflect on the effectiveness of their instruction.

EMSs support the use of instructional materials and students' multiple knowledge resources (e.g., mathematical, language, culture, family, interests) as a basis for relevant, meaningful, and responsive mathematics instruction. Some EMSs work within systems that have policies related to curriculum implementation fidelity and pacing requirements. At times these policies may pose challenges for responsive instruction, and EMSs help educators use their agency to productively navigate potentially conflicting messages. Finally, EMSs are mindful that learning and implementing new ways of teaching happens over time, and they provide ongoing and "just-in-time" support to educators as they try on new approaches and make shifts toward more ambitious and equitable practices.[103]

EMS as School-Level Mathematics Coach: When I began my coaching position straight out of the classroom, I had no idea how to address some of the instructional challenges my primary grade teachers were facing. After all, I had spent over a decade teaching at the 5th grade level. But, I could learn! I asked a 1st grade teacher leader if I could just sit in during her math time to learn more about student thinking at the primary level. Later that day, I talked with her about what I observed students saying and doing and listened carefully to everything she said. I then tactfully asked if she had any instructional needs that I could support. This "approach" got around to other educators at my school. Soon I began sitting in on mathematics lessons at the Kindergarten and 2nd grade levels, first always observing, followed by a conversation, and concluding with a query about how I might help. When it came time for me to plan for and implement professional learning sessions for my school, I engaged the teachers with whom I had first connected. I asked these early adopters to help me plan the sessions with their teams as well as co-lead presentations for the whole staff. All of the above helped in developing and strengthening relationships of trust and their confidence as teachers of mathematics.

Relevant Indicators: L.1.b., L.2.a., L.3.a., L.4.a.

L.2.b. Promoting asset-focused use of formative and summative assessments and data

EMSs recognize the importance of noticing and naming the strengths students possess and demonstrate through classroom-based formative assessment. With an interest in documenting these assets, EMSs potentially redesign assessments to draw out student thinking and invite multiple strategies to represent that thinking. They understand the importance of deeply understanding students' thinking, so instead of relying solely on formal written or computerized assessments, they support educators' use of observations,

[103] Gibbons & Cobb, 2017; Horn & Garner, 2022.

one-on-one conversations, and whole-class discussions as additional evidence of student understanding (i.e., content understandings, strategies, practices and processes).[104] EMSs assist with the systematic documentation and reflection on information gathered through both formal and informal assessments and partner with educators to collectively consider what they know about students' understandings and what they may still need to learn.

EMSs support educators as they responsively use information gathered through formative and summative assessments, which may include tests, quizzes, performance assessments, portfolios, and student reflection on their learning and growth (see Pedagogy Standard P.4). EMSs recognize the power of collaborative analysis of student work samples and use such work to support continuous improvement in instruction.[105]

L.2.c. Using summative assessment data for mathematics program improvement

EMSs are positioned to engage in grade level and potentially school- and district-wide decision-making on the use of summative assessments, including assessments developed by the district as well as those created externally (e.g., benchmark assessments, state-mandated assessments, progress monitoring tools). EMSs understand the problems with using language like “learning gap” and “achievement gap” and instead position observed differences as an “opportunity gap.”[106] Using asset-based views, they participate in their school’s or district’s analysis of summative assessment results, make appropriate interpretations, and deliberately frame dissemination and communication with teachers, families, and community members in clear and understandable ways.

EMSs use student summative assessment data to evaluate and modify mathematics programs at a variety of levels, including grade, school, and district. They understand the limitations and biases of summative assessments and strive to use and support others in using tools and strategies that are ethically grounded and developmentally, culturally, ability, and linguistically appropriate to document developmental progress and promote positive outcomes for all children.[107] Further, they understand the importance of assessment validity and reliability. They also understand and will challenge, as necessary, existing structural barriers grounded in assessments that limit and exclude children’s access and opportunities to learn meaningful mathematics, including high-stakes standardized assessments and readiness measures that lead to the labeling and sorting of children, resulting in segregation, marginalization, or privilege.[108]

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[104] Rigelman & Duden, 2023.
[105] Sherin et al., 2011.
[106] Ladson-Billings, 2006; Lubienski & Gutiérrez, 2008.
[107] National Association for the Education of Young Children, 2019.
[108] NCTM, 2020.

EMS as District-Level Mathematics Leader: As a division mathematics leader, one of my roles is to analyze data collected from our division-wide common assessments. Using the data provided by the testing program and organizing the data in a spreadsheet helps me to analyze the information and identify areas for further growth. This past year, we noticed that our students would benefit from extra support with the flexible use of computational strategies. We decided to plan professional learning focused on computational strategies that extend from operations with single-digit, to multi-digit, and ultimately to fractions and decimals. We intentionally scheduled the sessions about 2 weeks prior to the start of the unit for any given grade level so teachers have timely information they can immediately enact. Additionally, we noticed that students needed extra support in seeing the relationships between models and strategies, so when developing our unit guides for this year, we added information showing the progression as well as resources and lessons for teachers to use with their students. Using these data helps me to evaluate student growth and needs and to modify our program based on what our students could use rather than trying to guess.

Relevant Indicators: C.2.b., C.2.c., C.2.d., L.1.b., L.2.b., L.2.c.

Standard L.3. Activating Continuous Mathematics Professional Learning and Program Improvement

L.3.a. Assuming responsibility for their own ongoing learning

L.3.b. Designing and facilitating educator professional learning

L.3.c. Advocating for structures in schools that support ongoing educator learning and consistency in practice

Enhancing opportunities for ongoing mathematics-focused professional learning (PL) is a critical element of the rationale for developing and providing school- and district-based Elementary Mathematics Specialists (EMSs). As EMSs develop and navigate relationships with teachers, school- and district-level administrators, family and community members, and others, they are well-positioned to advocate for responsive and impactful PL for individual teachers or groups of teachers.

Like all other professionals in the field of education, EMSs themselves have learning needs. These needs are both similar to and different from the PL needs of teachers. EMSs must advocate and take responsibility for their ongoing growth in mathematics content and pedagogical knowledge, as well as the continued nurturing and development of their leadership knowledge and related skills. They need access to local, regional, and national professional organizations and groups, including memberships and conference attending opportunities. Finally, EMSs must stay abreast of policy initiatives that may impact their roles and responsibilities.

L.3.a. Assuming responsibility for their own ongoing learning

EMSs must take responsibility for their ongoing learning, growth, and identity development. This includes a focus on deepening their own understanding of mathematics content and pedagogy as well as knowledge, skills, and identity as a leader. Like educators, EMSs benefit from co-planning, co-teaching, coaching, and "doing the math" with one another, concurrently focusing on supporting educator learning.[109] They should also engage in analysis of their leadership practices by collecting evidence over time from students and educators to determine the effectiveness of their work.

Locally- or regionally-developed EMS PL may range from opportunities to meet with other EMSs to share experiences, needs, successes, and challenges, to more formal opportunities focused on particular responsibilities of an EMS. Other ongoing PL experiences include those provided by professional mathematics education organizations (e.g., National Council of Teachers of Mathematics, National Council of Supervisors of Mathematics: Leadership in Mathematics Education, Association of Mathematics Teacher Educators, TODOS Math for All) and their affiliates. Finally, EMSs should access and analyze research focused on effective and equitable practices and structures, and take action on issues related to mathematics,

[109] Borko et al., 2014.

instruction, curriculum, assessment, and policy impacting the learning of students, teachers, and others.

EMS as District-Level Mathematics Coach: I am one of three mathematics coaches in a very rural district. Each of us has been assigned to two of the district's six elementary schools. Our major responsibilities include building-based coaching, typically involving co-planning and lesson reflection and the connection of our coaching with PL opportunities for teachers. It took us close to 3 years to learn that we really needed to advocate for our own PL needs. We have acted. My two EMS colleagues and I, with enthusiastic approval from our mathematics supervisor, find time to meet one afternoon a week, typically each Friday from 1-3, to discuss teacher learning needs and related challenges, and just share ideas and discuss our learning needs. Sometimes it is a mathematics topic (e.g., connecting models and numeric representations to algorithms, supporting sense making and reasoning when problem solving), other times it is a teaching-related topic (e.g., the importance of questioning, strategies for engaging students in discourse, using mistakes as learning opportunities). We often discuss and share particular coaching or co-planning ideas that have worked well plus those that did not. More recently, we have been discussing our district's implementation of the state's revised mathematics standards and our collaboration with the Special Education team in developing our district's mathematics intervention program. Our sharing time, whether face-to-face or virtual and in- or out-of-school, provides all of us with time to reflect on our needs and challenges each week, and is so important to us. We truly learn from each other.

Relevant Indicators: L.1.b., L.3.b., L.4.a.

L.3.b Designing and implementing educator professional learning

EMSs working either as informal or formal leaders are called upon to design and implement PL for a range of educators (e.g., classroom teachers, resource teachers, paraprofessionals). They interact with both educators and school- and/or district-based administrators to determine PL needs. Based on these needs, EMSs establish PL goals that guide the planning and implementing of responsive, well-connected PL. This PL is internal to the school or district, and may also be external, such as conferences and university courses to increase local mathematics leadership capacity.

EMSs also recognize that productive PL may take a variety of forms and be offered to varying numbers of recipients .[110] When supporting individual educators, EMSs may provide resources (e.g., professional readings, research, frameworks, tasks, problems, routines); co-plan, co-teach, or model instruction; and engage in a coaching cycle (i.e., plan, gather data, reflect). Support for groups of teachers may include activities such as "doing the math" (i.e., engaging

[110] Gibbons & Cobb, 2017; Koellner et al, 2011.

with mathematical tasks), examining student work, analyzing classroom video or written cases, immersing in a book or article study, engaging in action research, and participating in lesson study or a modified version. Whether supporting individual teachers or groups of teachers, EMSs focus on deepening educators' understanding of and relationship with mathematics, along with improving students' learning experiences through school-wide implementation of ambitious and equitable mathematics teaching practices.

L.3.c. Advocating for structures in schools that support ongoing educator learning and consistency in practice

To achieve consistency in ambitious and equitable instruction and assessment, EMSs advocate for and help establish, maintain, and evaluate ongoing, high-quality mathematics PL opportunities.[111] They do this by collaborating with administrators to: 1) develop a PL plan that provides time for collegial learning dedicated to mathematics during regular grade-level team meetings, faculty meetings, or other structured release times; and 2) communicate expectations for participation in ongoing mathematics-focused PL. EMSs may also provide optional, outside-the-school day PL experiences, all with the intent of connecting educator learning to desired shifts in classroom practices. Finally, EMSs advocate for their own schedule to support cycles of co-planning and co-teaching with individual teachers or groups of teachers, as well as to plan for and implement school-wide mathematics PL.

[111] c.f., Darling-Hammond et al., 2017; Sztajn et al., 2017.

Standard L.4. Developing and Sustaining a Culture of Collaboration to Support Mathematics Teaching and Learning

L.4.a. Using knowledge of adult learners and learning to design professional learning

L.4.b. Cultivating relationships and sustaining collaboration with families and community members

Elementary Mathematics Specialists (EMSs) recognize that impactful professional learning is often collaborative. They use skills and strategies to work with a variety of adult learners and to achieve multiple goals. They implement techniques, such as the establishment of community agreements and use of protocols, to support groups in reaching agreement to the degree possible, while honoring diverse points of view. EMSs adapt their facilitation based on the size of the group, the familiarity the group members have with one another, and the length of time working with the group. They use effective listening and communicating strategies to develop and navigate relationships as they support groups in accomplishing tasks. They understand the culture and backgrounds of group members and encourage and value all perspectives and contributions.

L.4.a. Using knowledge of adult learners and learning to design professional learning

EMSs understand the principles of adult learning and know how to develop a collaborative culture of collective responsibility as they work within grade level, school, or district settings. They use this knowledge to promote a professional learning (PL) environment of collegiality, trust, openness, and respect that centers on continuous improvement in instruction and student learning. EMSs also recognize PL must actively engage educators while giving them choice and voice in the learning they will pursue.[112]

Due to their proximity with teachers and situated understandings, EMSs are well-positioned to offer school- or district- level mathematics PL opportunities, such as those focused on deepening understanding of (1) mathematics content such as operations with whole numbers and fractions; (2) mathematics practices and processes such as translation between contexts and more abstract representations; (3) responsive pedagogy such as supporting language development for multilingual learners or planning for students with identified learning needs in mathematics. EMSs work alongside teachers and have deep knowledge of teachers and their contexts. They are responsive to teachers' needs and facilitate meaningful PL, and engage adult learners in critical conversations to help improve the mathematics learning experiences of each and every student. Just as they would for students, EMSs recognize, highlight, and utilize teachers' strengths and cultivate teacher leadership.

[112] Horn & Garner, 2022; Kazemi & Hubbard, 2008.

EMS as School-Level Mathematics Leader: As a new EMS at my elementary school, I was warned by the administrative team that the teachers rarely actively participated in professional learning (PL) sessions and typically preferred to work alone. I always like a good challenge, so I consulted with my district's mathematics coordinator for suggestions. The first few weeks of school, I used my planning period to get to know each teacher by visiting their classrooms, working with students, and providing them with an interest survey to find out their needs, preferences, and areas of concern. I also provided my colleagues with a menu of possible offerings to identify the topics and types of PL activities that would be of interest to them. To honor their voices, I used the information gathered to plan upcoming PL sessions. I was able to: (1) provide teachers with choices of PL sessions they could attend based on their needs and interests; and (2) offer a variety of formats such as morning Coffee Chats, Pizza Thursdays, and Lunch & Learn sessions to accommodate their varied schedules. I was intentional about including opportunities for active learning and collaboration in every session. These efforts helped me to not only learn about teacher learning needs but also allowed me to identify teacher leaders and early adopters of the approaches learned. I worked with these emerging leaders to create a mathematics committee consisting of a teacher from each grade level. This committee met on a regular basis to plan school-wide initiatives like Mathematics/STEM night, plan PL sessions, select professional books for our faculty book study, and advocate for standards-aligned mathematics at our school.

Relevant Indicators: L.3.b., L.4.a.

L.4.b. Cultivating relationships and sustaining collaboration with families and community members

EMSs understand that the cultures and backgrounds of families and communities play a significant role in cultivating and improving students' mathematics dispositions, learning experiences, and outcomes. EMSs recognize the importance of working through their own biases, understanding others' experiences and intersectional identities, and using their knowledge to strengthen partnerships with families and community members. They also know the importance of reciprocal family-community-school partnerships and the need to create a shared vision of and responsibility for deep, meaningful, relevant mathematics learning for all students.[113] Finally, EMSs work with colleagues to promote ongoing systematic communication and collaboration with families, community members, business and community leaders, and others to improve the educational system and expand opportunities for students.

[113] Safir & Dugan, 2021.

EMS as Specialist Teacher of Mathematics: Each year we host Family Math Experiences for our students and their families. What was once called Family Math Nights, became Family Math Experiences when we recognized we could reach more families if we offer both the option of an evening event as well as an event just after drop off in the morning. While these events are well attended and successful, many families want to know what they can do to help their children with mathematics on a regular basis. Our current curriculum has weekly newsletters, available in English and Spanish, to describe the topics for the week and provide tips for completing homework. We receive mixed reviews from families about their use of the newsletters at home. The mathematics specialist teachers for each grade level met, and we decided to create short videos each week called *Math in a Minute.* The *Math in a Minute* videos provide families with a quick overview of the mathematics concepts being covered in class that week by grade level, along with visual representations of the concepts or strategies. These quick snapshots of mathematics concepts have been a big hit with families. We use multiple ways to disseminate the videos (e.g., Class Dojo, QR codes, email, texts), so we are reaching a larger percentage of our families. Next year we hope to make both English and Spanish versions of the videos available to families.

Relevant Indicators: C.math storylines., P.3.b., L.4.b.

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For the complete list of standards and policy guidelines as well as resources to support EMS programs and professional learning development go to: http://amte.net/ems.

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